MANUAL

OF

PLANE GEOMETRY,

ON THE HEURISTIC PLAN,

WITH NUMEROUS EXTRA EXERCISES, BOTH THEOREMS AND PROBLEMS, FOR ADVANCE WORK.

BY

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PREFACE.

This book is published primarily for the author's pupils, and secondarily for that constantly increasing number of teachers who are getting more and more dissatisfied with the old methods of teaching geometry, but who have hitherto found no manual suited to their needs. That the reasoning faculties of a child or youth are developed by use, goes without saying; and consequently the method of teaching geometry whereby the pupil *originates* the demonstration, or simply *demonstrates*, rather than *memorizes* the demonstrations of another, needs no defence at my hands. This manual has stood the test of three years' work in the class-room, and differs from other geometries that provide for original work, in that the original demonstrations and constructions are not *side issues*, so to speak, but are required of the pupil in that sequence of theorems and problems which may be called the regular required work, and of which complete demonstrations are usually given in other manuals for the pupil to memorize. In this work demonstrations are given only where the average pupil would be at a loss to know how to proceed, and generally, as illustrative of methods, while others are partially given and left for the pupil to complete. In other cases, wherever the author has found that his own pupils were not working to advantage, he has introduced suggestions, of which pupils may or may not avail themselves.

The old method of division into *books* has been abandoned, as it serves no practical use, since different authors have made different divisions. It serves to make the subject a continuous one in the mind of the pupil, without the artificial breaks of the old way.

The arrangement of the theorems whereby the essential ones, together with several simple additional ones for giving the pupil more drill, followed by a set of non-essential but more difficult ones for advance work, the author has found advantageous.

He has also found it more useful to the pupil to be compelled to construct his own diagram, and state the *converse* of the theorems he is to prove. Practical problems of computation have also been given, immediately following any given subject, so that the pupil can immediately see the practical application of the theorems he has demonstrated.

All the problems of construction have been placed together after the theorems, for the sake of uniformity, as they form no part of the logical sequence of geometrical truths as embodied in the theorems. Many of them are of practical, as well as disciplinary, importance, however, and so it is left to the skilful and judicious teacher to take them up as the interests of his pupils demand.

The author has received valuable suggestions from Professor E. L. Richards of Yale College, and also from Professor T. H. Safford of Williams College. He also desires to publicly express his indebtedness to Hon. J. W. Patterson, Superintendent of Public Instruction for New Hampshire, and E. R. Goodwin, Principal of the High School at Lawrence, Mass., for their hearty and outspoken indorsement of his

work, and encouragement to persevere in elaborating the method.

Finally, thanks are due the publishers and printers for the excellence and beauty of the mechanical work.

In conclusion, the author would be glad to receive suggestions from those teachers into whose hands this manual may chance to fall, with a view to its improvement as a regular class text-book.

G. I. H.

MANCHESTER, N.H.,
May, 1891.

INTRODUCTION.

GEOMETRY is highly important, and growing in importance as a branch of mathematical study. This is not only true for mathematicians; but, what is extremely interesting to teachers, to practical men also. The vast development of machinery, of steam-power, of the applications of electricity, bring chapters of mathematical science into every-day use which have long been employed in physical investigation, but never before in the arts of life; and in the construction of all kinds of machinery and industrial devices the geometrical representation to the eye is of far more immediate and practical value than abstract calculations. The men who can draw are rapidly gaining on those who can merely calculate. In such matters as statistics the old processes are even reversed. We have not only an application of arithmetic to geometry, but geometrical representations of arithmetical results; there is not merely an algebraic geometry, but a graphic algebra.

In manual training schools geometry is fully as important a branch of mathematics as arithmetic, even for the future mechanic. The great defect in American mathematical training has been that arithmetic and algebra have been too much favored as against geometry. Teachers have delayed, and still delay, presenting even the elements of geometry till a great deal of algebra has been mastered; the meagre facts which must be stated before even mensuration can be intelligently treated have been reduced to the smallest compass and the most mechanical shape; and it is only because we confuse the difficulty of the subject with our stupid ways of teaching it that we tolerate the geometrical ignorance of our pupils.

There are skilful mathematicians who are unaware how much better geometers early training would have made them, if the geometrical side of things had had fair play in their education. But a reform is impending; and Mr. Hopkins's text-book here presented is intended to promote it. I desire to call the attention of all earnest and progressive teachers to the heuristic method as here expounded.

The word *heuristic* is derived from the Greek; it means the method of discovery. *Inventional* is another word which has been somewhat similarly used; but, I think, in a narrower meaning.

In mathematics the "heuristic" is the same as the "development" method; that is, the method by which the pupil is led to see the theorems and their demonstrations for himself.

In attempting to use this book, the ordinary laws of good teaching must be followed. Consequently the pupils, however mature, must possess all the prior qualifications; they must be intelligent as to the subject-matter. The greatest difficulty in now teaching geometry by any method lies in that neglect of the elements to which I have before alluded. If a pupil reaches the age of fifteen (as the pupils of the Massachusetts grammar schools are said to do) without thorough and systematic training in the elements of the subject, nothing remains but to prefix a course of instruction in these elements to the study of demonstrative geometry. A very skilful and celebrated teacher was about 1875 mentioned to me by his pupils as using the heuristic method; and I at once wrote him to inquire how he presented the elements. His answer was that about six weeks were spent in working up, orally, the doctrine of form; that important part of geometry whose results are contained in the definitions and other preliminary matter. This was in America, with pupils who had passed the grammar school without learning these elements; and represents what seems to the writer to be the minimum of attention to be given to this part of the subject.

On the other hand, the Austrian higher schools spend four years (one and one-half to two hours weekly) in that empirical form of geometry in which attention is given rather to acquiring a knowledge and practical use of the subject (both plane and solid) than to its logic as a strictly scientific branch of study. These boys are from eleven to fifteen years old, on the average, and correspond to our grammar school pupils in age and general maturity. Their instruction, so far as I can judge from the text-books and the books of direction to the teachers, is almost entirely heuristic in character; and a former pupil of mine of Austrian birth (who is now an eminent American professor of an ancient language) confirmed this, and assured me that then (twenty years ago) this method was actually employed in his training.

In the earlier stages of such a course, immediate inspection shows the simpler geometrical truths with full conviction and ready acceptance; precisely as the pupil learns that seven times nine equals nine times seven, not by algebraic demonstration (which involves several steps), but by practical experience. Mathematicians in the higher branches make great use of intuitional proofs, supplemented if need be by logical demonstration; and in the lower the same law holds good on the geometrical side as well as on the arithmetical.

The beginner in geometry should not at first be required to perform all the geometrical operations at once, nor with great rapidity. The careful series of objective illustrations of geometrical form, combined with practical exercises and simple reasonings, cannot, under our present programme, receive all the expansion which it has in Austria. Our teachers are compelled to deal with their pupils in a short time, and to teach them to geometrize in comparatively few lessons; and so the time to practically carry out the heuristic method must be obtained by economizing opportunities.

But the old-fashioned method of memorizing the whole book, definitions, propositions, corollaries, scholia, even the numbers

which were prefixed to these truths to indicate their order, is happily on the decline. It is a wasteful method, and communicates far more information than can be permanent or useful; and it lays a heavy burden on the memory. When the heuristic method takes its place, the pupil will first of all be brought, by successive steps of abstraction, from the study of geometrical models, to the idea of geometrical solids, surfaces, lines, points; they will then be taught the generation of lines by the motion of points, the various classes of lines, the simplest figures bounded by straight lines, triangle, square, rectangle, rhomboid, trapezoid, and trapezium. The analysis should be directed to the cube, the other four regular solids, the ordinary solids bounded by straight lines. From these the idea of plane angles in their various classes should be obtained; the difference and relationship between plane and diedral angles should be insisted upon. In what does an angle on the blackboard differ from that formed when a book or a door is opened?

The doctrine of form should be extended somewhat beyond the things mentioned in plane geometry; the reciprocal relation between the cube and the octaedron (each has as many faces as the other has vertices; the number of their edges is equal) should be pointed out on the model and exemplified by drawing one inscribed in the other. Similar relations exist between the dodecaedron and icosaedron; and two tetraedrons. Again, in analyzing all convex solids bounded by plane faces the fact should be brought out that $E+2=F+V$; that is, the cube has 12 edges, 6 faces, 8 vertices; $12+2=6+8$.

A skilful object teacher, with a few models, can thus readily develop a great many interesting truths, and thus prepare the young people's minds for geometry; a few weeks' time can be well spent on the definitions.

Euclid's text-book, which is on a different plan from the modern books, begins rather more gradually; and our best teachers imitate him in going more slowly over the early prop-

ositions. It seems to me that the models bounded by curved surfaces (cylinder, cone, sphere, frustum of a cone, and the parts of a cone bounded by the sections, as well as the ellipsoid) ought to be exhibited and so far analyzed as to furnish material for the definitions of straight lines and plane surfaces, in distinction to curved lines and surfaces. And I would go a step farther; and show the difference between the ruled surfaces (like the cylinder and the cone) and those in which no straight lines can be drawn.

In a word, I would lay the foundation for solid geometry (which is the practical form of the science) along with plane, which is a mere abstraction from the other. The introduction to demonstration may well be combined with additional study of form. Let the pupils, for instance, be taught to make triangles; then to measure the sides and angles of those they make; then to classify their triangles into equilateral, isosceles, scalene; equiangular, triangles with two equal angles, with no equal angles; then to set down a few theorems they may expect to prove. This is no doubt the way the Egyptian priests built up the propositions into which, as the foundations of a secret society, they initiated Pythagoras. Right angles, vertical angles, exterior angles, can now be brought in naturally; then triangles with a short side and two long ones; then the idea of parallels. Any bright teacher who is possessed with the desire of training pupils to think geometrically will naturally fall into the line of thinking needed; but it all takes time.

I estimate that, when the definitions and those matters which make up the first book of our ordinary geometries have been fully mastered, half the work of the course in plane geometry has been accomplished; and the teacher must bear in mind that if this is well done, the pupil has been trained, not crammed, to a very high point in the subject.

The modern writers on pedagogy lay down the principle that interest is the end of teaching, and not merely the means.

If you have brought your pupil to enter upon mathematics with an earnest desire to accomplish the work in it, you have done more than if you merely cram it into him. In the colleges it is well known that the great mass of students do comparatively little in the science. When they have the choice, they prefer as a rule to take something easier. Just so most boys never accomplish much in their athletics. The fashion has come about that about a quarter of our students play ball, more or less, and the rest look on; simply because their muscles have not been trained to the work. Now what we want in mathematics is precisely analogous,—training of the mathematical muscles. The German use of "gymnasium" as the name for those high schools which do the work preparatory to the universities — schools which accomplish nearly as much at the age of nineteen or twenty as our colleges do three years later on — is in the same line of thinking. Let us make our high schools true gymnasia; and a good subject to begin with is geometry.

What, then, is the true exercise of the mathematical powers? (I had almost said of the mathematical muscles.) Grube's numerical analysis of the numbers from 1 to 100 is the heuristic method in arithmetic; the pupils are led on by the teacher to analyze and find out arithmetical truths for themselves. It is much to be wished, as Pestalozzi taught, that form might be thoroughly taught in the primary school; Steiner, whose geometrical ideas lie nearly at the foundation of modern developments, was a pupil and teacher of Pestalozzi's own school in Switzerland. But if this is not yet practicable, let the pupil take up geometry from the beginning at a later age; and really do fundamental and thorough work.

The teacher, in using the heuristic method, must be very careful of his foundations; he must develop, and thoroughly develop, his definitions, instead of giving them out as a task. The first demonstrations must be slowly and carefully done; neither the Egyptian priests nor Pythagoras discovered the early theorems all at once, or in any short time.

The hill of mathematical knowledge is high and steep, and few go up on it any great distance. There are three ways of ascending. The pupil may be carried up in a wagon; this is tolerably easy, and there is some gain of fresh air and a wide prospect. It is like the committal to memory of a text-book. He may be dragged up as tourists sometimes ascend the Alps; tied to a rope with a strong guide at either end. This is still more beneficial; but the best way of all is to be trained to walk up for himself. At first, the trainer must content himself with gradual development of the muscles, and little apparent ascent; but the continual exercise finally enables him to reach a great height with comparative ease.

In preparing pupils to pass college examinations the heuristic method has a great practical value. Once the method of making a demonstration is known and the few data upon which the questions depend are learned, the student who can demonstrate originally has a great advantage over him who relies more largely on his memory. If the latter forgets, he is lost; but the former can originate when memory fails, and if he has been well trained is far less liable to lose his presence of mind.

I congratulate Professor Hopkins on the spirit with which he has undertaken to fill a gap in our geometrical literature; and I hope the book will have the success which it deserves.

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August, 1891.

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APPENDIX.

BOOK I.

PLANE GEOMETRY.

1. *Space* is indefinite extension in every direction.

2. A *material substance* is anything, large or small, solid, liquid, or aeriform, visible or invisible, that occupies a portion of space.

3. It therefore follows that material substances have *limited extension* in every direction.

4. For purposes of measurement, extension in *three* directions only are considered, called, respectively, *length*, *breadth* (or width), and *thickness;* they are also called collectively *dimensions.*

5. *Magnitude,* in general, means size, and is applied to anything of which greater or less can be predicated, as time, weight, distance, etc.; a *geometrical magnitude* is that which has one or more of the three dimensions.

6. A *geometrical point* has position merely; i.e. it has no magnitude.

The dots made by pencil and crayon are *called* points, but they are really small substances used to indicate to the eye the location of the geometrical point.

7. A *geometrical line* has only one dimension; i.e. length.

The lines made by pencil and crayon are substances, and

may be called *physical lines* which serve to show the position of the geometrical lines.

8. A *straight line* is one that lies evenly between its extreme points.

This is the definition as given by Euclid. The majority of modern geometers, however, have substituted the following as stated by Newcomb; viz.:

"A *straight line* is one which has the same direction throughout its whole length."

Each is designed to *express* the idea of straightness, and not to *convey* it, for it is assumed that the idea already exists in the pupil's mind prior to the beginning of this study.

9. A *curved line,* or simply *curve,* is one no part of which is straight.

10. Material substances have one or more faces which separate them from the rest of space. These faces are called *surfaces,* and have, obviously, only two dimensions; i.e. *length* and *breadth.*

11. The surface considered apart from the substance is called a *geometrical surface.*

12. A *plane* is a *geometrical surface* such that if any two points in it be selected at random, the straight line joining them will lie wholly in that surface.

13. A *curved surface* is a geometrical surface no portion of which is a plane.

14. A physical solid is the material composing it, and which we perceive through the medium of the senses; while the *geometrical solid* is the *space,* simply, which the physical solid occupies.

15. A *geometrical figure* is the term applied to combinations of points, lines, and surfaces, when reference is had to their form or outline simply.

16. A *plane figure* is one whose points and lines all lie in the same plane.

17. "A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line." — EUCLID.

"An *angle* is a figure formed by two straight lines drawn from the same point." — CHAUVENET.

"When two straight lines meet together, their mutual inclination, or degree of opening, is called an *angle*." — LOOMIS.

18. The lines which form an angle are called the *sides* of the angle, and the point from which they are drawn is called the *vertex* of the angle.

19. When two plane angles have the same vertex and a common side, *neither angle being a part of the other*, they are said to be *adjacent* angles.

20. When two angles have the same vertex and the sides of one are the extensions of the sides of the other, they are called *vertical* angles.

21. An angle is named by a letter or number placed at its vertex. If, however, there are two or more angles with the same vertex, other letters are placed at the extremities of their sides, and the three letters are used to name the angle, the letter at the vertex always coming between the other two.

22. Let us consider the point *B*, in the straight line *AC*, a pivot, and *BD* another starting from the position *BC*, and

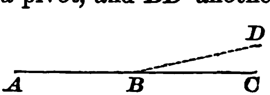


Fig. I.

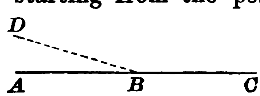


Fig. II.

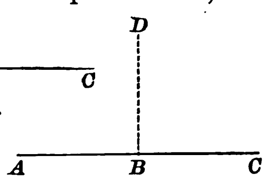


Fig. III.

turning about *B*, keeping always in the same plane. It is evident that, as soon as it has started, it forms *two angles* with the line *AC*, of which *DBC*, Fig. I., is the

smaller. If it continue to revolve, however, it will finally reach a position as *DB*, Fig. II., in which the angle *DBC* is the *larger.* Hence, in passing from the first position to the second, it must have reached a position, Fig. III., where the two angles *DBC* and *DBA* were equal.

23. Hence, when one straight line meets another so as to form equal adjacent angles, each of the angles is called a *right angle,* and the lines are said to be *perpendicular* to each other.

24. It is also evident that the sum of the angles formed by any one position of the line *BD* is equal to the sum of the angles formed by any other position; for what is taken from one angle by the revolution of the line *BD* is added to the other.

25. Hence, when one straight line meets another so as to form two angles, the sum of these two angles equals two right angles.

26. An angle that is less than a right angle is called an *acute* angle.

27. An angle that is greater than one right angle and less than two is called an *obtuse* angle.

28. Both acute and obtuse angles are designated as *oblique* angles as contrasted with right angles.

29. A *straight* angle is a term recently adopted by prominent English and German mathematicians to express, by a single unit, the sum of two right angles.

30. When the sum of two angles is equal to a straight angle, they are said to be *supplementary;* i.e. each is the supplement of the other.

31. When the sum of two angles is equal to one right angle, they are said to be *complementary;* i.e. each is the complement of the other.

32. It is evident from 22 and 23 that when one straight line meets another *so as to form two angles*, these angles are *supplementary*.

33. Two straight lines are said to be *parallel* when, lying in the same plane, and extended indefinitely both ways, they do not meet each other.

34. As we have before conceived a line (22) to move, so we may conceive one geometrical magnitude to be applied to another for the purpose of comparison. If they coincide, point for point, they are said to be *equal.*

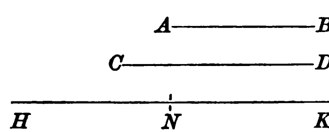
35. Thus, if two angles can be so placed that their vertices coincide in position and their sides in direction, two and two, the angles must be equal.

36. Conversely, if two equal angles be conceived to be so placed, one upon the other, that their vertices and one pair of sides coincide respectively, then the other pair of sides must also coincide, otherwise one angle would be greater than the other.

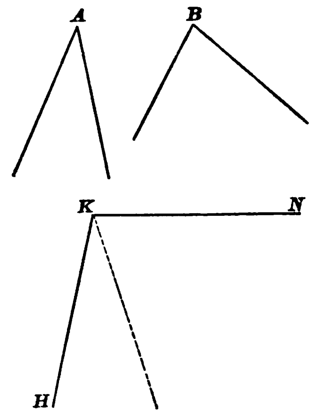
37. *Geometrical magnitudes* are geometrical *lines*, *angles*, *surfaces*, and *solids.*

38. We shall have occasion to express the addition and subtraction of geometrical magnitudes, as well as the multiplication and division of these magnitudes by numbers.

39. For example, the sum of the two lines *AB* and *CD* is obtained by conceiving them to be placed so as to form one continuous straight line as *HK*. Similarly, the difference of two lines is obtained by cutting off from the larger a line equal to the smaller. Similarly, *HKN* represents the sum of the two angles *A* and *B*. To multiply a line by a number is to add it to itself the required number of times. (See above.)



To divide a line by a number is to conceive the line to be divided into the required number of equal parts.



The same is true of other geometrical magnitudes. (Illustrations of each should be given.)

40. An *axiom* is a truth that needs no argument; i.e. the mere statement of it makes it apparent, e.g.:

GENERAL AXIOMS.

I. The whole of anything is greater than any one of its parts.

II. The whole of anything is equal to the sum of all its parts.

III. Quantities which are respectively equal to the same or equal quantities are equal to each other.

IV. Quantities which are respectively halves of the same or equal quantities are equal to each other.

V. Quantities which are respectively doubles of the same or equal quantities are equal to each other.

VI. If equal quantities be added to equal quantities, the sums are equal.

VII. If equal quantities be subtracted from equal quantities, the remaining quantities are equal.

VIII. If equal quantities be multiplied by the same or equal quantities, the products are equal.

IX. If equal quantities be divided by the same or equal quantities, the quotients are equal.

X. If equal quantities be either added to or subtracted from unequal quantities, the results will be unequal.

XI. If equal quantities be either multiplied or divided by unequals, the results will be unequal.

41. The results obtained by the addition to, subtraction from, multiplication or division of, unequals by unequals are indeterminate with one exception. The pupil should ascertain for himself this exception.

42. *Particular axioms.*

XII. Between two points only one straight line can be drawn; or if others are drawn, they must coincide.

XIII. A straight line is the shortest of all possible lines connecting two points.

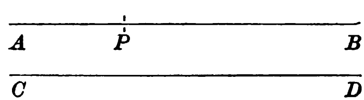
XIV. Conversely, the shortest line between two points is a straight line.

XV. If two straight lines have two points in common, they will coincide however far extended.

XVI. Two straight lines can intersect in only one point.

XVII. In one direction from a point only one straight line can be drawn; or if more be drawn, they must coincide.

XVIII. Through a given point (as P) only one line (as AB) can be drawn parallel to another line (as CD); or if others are drawn, they must coincide.



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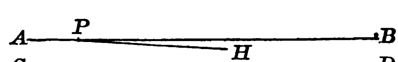
XIX. If a line makes an angle with one of two parallel lines, it will intersect the other if sufficiently extended.



XX. The extension or shortening of the sides of an angle does not change the magnitude of the angle.

43. A *theorem* is a truth which is made apparent by a course of reasoning or argument. This argument is called a *demonstration.*

Every theorem consists of two distinct parts, either expressed or implied; viz. the *hypothesis* and *conclusion.* The conclusion is the part to be proven, and the demonstration is undertaken only upon the ready granting of the conditions expressed in hypothesis; e.g.:

Hyp. If two parallel lines be crossed by a transversal,

Con. the alternate interior angles are equal.

44. In demonstrating the theorems in this book the pupil should first *analyze* the theorem and write it after the above model. For instance, let us analyze the following theorem; viz.:

A perpendicular measures the shortest distance from a point to a straight line.

Now this theorem, analyzed and written according to our model, would read as follows:

Hyp. If from a given point to a given straight line a perpendicular and other lines be drawn,

Con. the perpendicular will be the shortest one of those lines.

45. The *converse* of a theorem is another theorem in which the hypothesis becomes the conclusion, and conclusion the hypothesis. For example, the converse of the theorem mentioned in Sect. 43 would read as follows; viz.:

Hyp. If two straight lines in the same plane be crossed by a transversal so as to make the alternate interior angles equal,

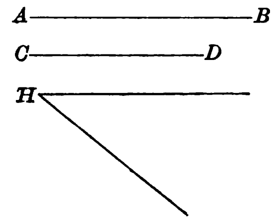
Con. these two straight lines will be parallel.

The converse of most of the theorems in this work will be left for the pupil to state.

46. A *problem,* in geometry, is the required construction of a geometrical figure from stated conditions or data; e.g.:

It is required to construct the triangle which has for two

of its sides *AB* and *CD*, and the angle *H* included between these two sides.



A *postulate* is a self-evident problem, or a construction to the possibility of which assent may be demanded or challenged without argument or evidence.

(Both theorems and problems are commonly designated as propositions.)

47. Before we can accomplish the demonstration of a theorem in geometry, the following postulates must be granted; viz.:

I. A straight line can be drawn from one point to any other point.

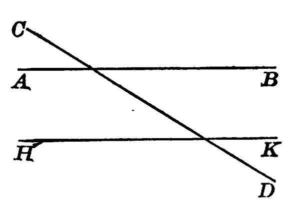
II. A straight line can be extended to any length, or terminated at any point.

III. A circle may be described about any point as a centre and with any radius.

IV. Geometrical magnitudes of the same kind may be added, subtracted, multiplied, and divided.

V. A geometrical figure may be conceived as moved at pleasure without changing its size or shape.

48. A *postulate* is also used to designate the first step in the demonstration whereby certain conditions or data are demanded as fulfilled, or admitted as true, for the basis of the argument, and which begins with "Let," etc., or "Let it be granted that," etc. It really demands assent to the general conditions implied in the hypothesis, with special reference to a particular representative diagram or figure; e.g. in beginning the demonstration of the theorem given in Sect. 43, we should say: "Let *AB* and *HK* be two parallel straight lines crossed by the transversal *CD*."



This is the postulate; i.e. it matters not whether *AB* and *HK* are actually straight or actually parallel; they ***stand for*** straight and parallel lines, and the argument is just as conclusive when based upon their ***supposed*** parallelism, as it would be if we had positive knowledge that those two identical lines were parallel.

49. The demonstration of the following theorems should be written by the pupils, with an occasional oral exercise participated in by the entire class, each member in turn contributing a single link in the chain of argument.

50. In order to save time for the pupil in writing and the instructor in correcting, the author, having used them, recommends the use of the following list of symbols and abbreviations, as well as such others as the instructor and pupils may agree upon:

51. **SYMBOLS.**

$+$	plus.	$\angle$	angle.
$-$	minus.	$\angle$s	angles.
$\times$	multiplied by.	rt. $\angle$ or $\lfloor$R	right angle.
$=$	equals, or is equal to.	rt. $\angle$s or $\lfloor$R's	right angles.
$\therefore$	therefore, or hence.	$\triangle$	triangle.
$>$	is greater than.	$\triangle$s	triangles.
$<$	is less than.	rt. $\triangle$	right triangle.
$\equiv$ or $\asymp$,	equivalent to.	rt. $\triangle$s	right triangles.
$\odot$	circle.	$\perp$	perpendicular.
$\odot$s	circles.	$\perp$s	perpendiculars.
$\parallel$	parallel.	$\square$	parallelogram.
$\parallel$'s	parallels.	$\square$s	parallelograms.

52. ABBREVIATIONS.

Adj.	adjacent.	Int.	interior.
Alt.	alternate.	Line	straight line.
Ax.	axiom.	Opp.	opposite.
Comp.	complementary.	Post.	postulate.
Con.	conclusion.	Prob.	problem.
Cons.	construction.	Pt.	point.
Def.	definition.	Quad.	quadrilateral.
Dem.	demonstration.	St.	straight.
Dist.	distance.	Sug.	suggestion.
Ext.	exterior.	Sup.	supplementary.
Hyp.	hypothesis.	Trans.	transversal.
Iden.	identical.	Vert.	vertical.

Q.E.D. . . . Quod erat demonstrandum.

Q.E.F. . . . Quod erat faciendum.

53. The last two expressions are in Latin, and mean respectively, "which was to be demonstrated" or "proven," and "which was to be performed" or "done." The former is placed at the close of the demonstration of every theorem to indicate that the required proof has been completed; while the latter is placed at the close of the work of every problem to indicate that the required construction has been completed.

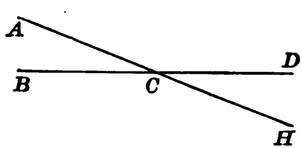
54. The pupil must remember that *every statement* in geometrical demonstration must be "backed up" or substantiated by giving as authority, definitions, axioms, and previously established truths, and *every unsupported statement,* whether from instructor or fellow-pupil, should be promptly challenged.

55. Complete demonstrations are given of only a few of the following theorems, and those solely for the purpose of giving the pupil a start when encountering a new subject, or in places where the pupil's time would not be spent to advantage in his helpless work; i.e. where the average pupil would be at a loss to know how to go at the subject. In

other places occasional suggestions are made for the purpose of giving the pupil a start without loss of time. In only a few cases have any diagrams been given, the author thinking it best to leave them as a part of the pupil's exercise. And right here he wishes to emphasize this caution to both instructor and pupil, based on several years' experience in the class-room; viz.: *Always employ the most unfavorable diagram.* Care in this particular will save many an inadvertent error, and prevent the *assumption* of conditions unwarranted by the hypothesis.

THEOREMS.

56. *Hyp.* If one straight line intersects another,
Con. the vertical angles are equal.



Post. Let AH and BD be two straight lines intersecting each other at point C.

We are to prove (1) $\angle ACD = \angle BCH$ ~~(I.)~~

and (2) $\angle ACB = \angle DCH.$ ~~(II.)~~

Dem. $\angle ACD + \angle ACB = 2$ rt. $\angle$s. (Sect. 32.)

$\angle BCH + \angle ACB = 2$ rt. $\angle$s. (Same reason.)

$\therefore \angle ACD + \angle ACB = \angle BCH + \angle ACB.$ (Axiom III.)

$\angle ACB = \angle ACB.$ (Iden.)

$\therefore \angle ACD = \angle BCH.$ (Axiom VII.)

Q.E.D.

The pupil should now employ a similar method and demonstrate (II.) and the following easy ones:

57. If two straight lines intersect each other, the sum of the four angles thus formed equals four right angles.

58. The sum of all the angles formed at one point on the same side of a straight line equals two right angles.

59. The sum of all the angles formed by any number of straight lines meeting at one point equals four right angles.

Sug. Extend one of the lines.

Remark. In the demonstration of most theorems it becomes necessary to draw one or more auxiliary lines in addition to those involved in the original figure. These are called, technically, *construction lines,* and should always be *dotted,* in order that they may readily be distinguished from the *given* lines, or those comprising the original figure.

60. If one of the four angles formed by the intersection of two straight lines is a right angle, the other three angles are also right angles.

61. All right angles are equal.

62. If two angles be equal, the complements of those angles are also equal.

63. If two angles be equal, the supplements of those angles are also equal.

64. If two supplementary adjacent angles be bisected, the bisecting lines are perpendicular to each other.

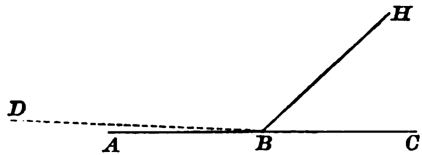
65. From one point in a straight line only one perpendicular to that line can be drawn in the same plane; or if more be drawn, they must coincide.

Sug. Consult Sects. 22, 23, and 36; also Theorem 61.

66. *Hyp.* If two adjacent angles be supplementary,

Con. their exterior sides form one and the same straight line.

Post. Let ABH and HBC be 2 sup. adj. $\angle$s. **We are to prove that AB and BC form one and the same st. line.**



Dem. BC is, by *Hyp.*, a st. line, therefore if it be extended in either direction, it will still be a st. line. If now we can prove that the extension of BC must coincide with AB, then AB and BC must form one line.

There are only two possible relations, as regards *position*, between AB and the extension of BC; i.e. they either coincide or they do not. Now, if we can prove the impossibility of *all* the *possible* relations except one, whether two or more, knowing that *one* of them *must be true*, that is equivalent to a direct proof of the existence of that one relation, and is just as conclusive.

For example, we know that of the two quantities x and y, one of three relations must be true; viz.: (1) $x > y$, (2) $x = y$, (3) $x < y$. There is no other possible relation in respect to size; hence, if we can prove the impossibility of the first two relations in any given case, then that is conclusive that the third is true, and that consequently $x < y$.

Let us apply this method to the demonstration of this theorem.

It is evident that AB is either the extension of BC or it is not. Let us suppose that the latter relation is true and then trace the consequences. If AB is not the extension of BC, then some other line, as BD, must be. Now, if BD is the extension of BC, *CBD must be a st. line,* whether it seems to the eye to be so or not.

$$\therefore\ \angle HBC + \angle HBD = 2 \text{ rt.} \angle s, \qquad \text{(Sect. 32.)}$$

but $$\angle HBC + \angle HBA = 2 \text{ rt.} \angle s; \qquad \text{(By Hyp.)}$$

$$\therefore\ \angle HBC + \angle HBD = \angle HBC + \angle HBA. \quad \text{(Axiom III.)}$$

$$\angle HBC = \angle HBC; \quad \text{(Iden.)}$$

$$\therefore\ \angle HBD = \angle HBA. \quad \text{(Axiom VII.)}$$

Let us examine this last equation. Can the equation be a true one? Can the $\angle HBD$ be equal to the $\angle HBA$? Evidently not, for the former is a *part* of the latter (Axiom I.). We have, therefore, arrived at an *impossible* result. It is plain, therefore, that there must be error somewhere; either in the argument or in the *supposition* upon which the argument is based. A careful and critical review of the argument discloses no flaw, no error, but, on the contrary, it is in perfect accord with previously established principles; consequently the conclusion is inevitable that the error is in our supposition. If, then, our supposition that BA is not the extension of BC is erroneous, the only other possible relation *must be true;* viz. that BA is the extension of BC, and therefore they must form one and the same straight line. Q.E.D.

67. If two vertical angles be bisected, the bisecting lines form one and the same line.

Sug. Use Theorems 64 and 65.

68. If two pairs of vertical angles be bisected, the bisecting lines are perpendicular to each other.

Sug. Consult Theorems 64 and 67.

69. The line which bisects one of two vertical angles will, if extended, bisect the other.

ANGLE MEASUREMENT.

70. The most common unit for measuring the magnitude of angles is the ninetieth part of a right angle, called a *degree.* The degree is subdivided into sixty equal parts called *minutes*, and the latter also into sixty equal parts called *seconds.* By

this means small fractions of a degree may be expressed in minutes and seconds. These units are indicated by the following symbols; viz.:

$^\circ$ for degrees, $'$ for minutes, and $''$ for seconds;

e.g. $75^\circ\ 20'\ 34''$.

This is called the *sexagesimal* method. Another method, called the *centesimal*, has been proposed in France, having obvious advantages. But owing to the fact that all mathematical tables and instruments had been arranged and constructed with reference to the sexagesimal method, it has not yet come into extensive practical use.

In this method the right angle is divided into *one hundred* equal parts called *grades*, the grade into the same number of equal parts called *minutes*, and the latter into *one hundred* equal parts called *seconds*. These are designated by symbols as follows; viz.:

$45^g\ 15^{\backprime}\ 28^{\backprime\backprime}$.

How many degrees in a right angle?

How many grades in a right angle?

What is the complement of a right angle?

What is the supplement of a right angle?

How many degrees in all the angles formed at one point on one side of a straight line? How many grades?

How many grades in all the angles formed by any number of straight lines meeting at a common point? How many degrees?

What is the complement of 45°; 30°; 1°; $89^\circ\ 59'\ 59''$; 45^g; 30^g; 1^g; $89^g\ 59^{\backprime}\ 59^{\backprime\backprime}$?

What is the supplement of each of the above angles?

How many degrees in an angle that is double its complement? How many grades?

How many degrees in an angle that is four times its supplement? How many grades?

What angle, in degrees and grades, do the hour and minute hands of a clock form at 2 o'clock? At 3 o'clock? At 5 o'clock?

If one of the angles, formed by two straight lines crossing each other, be 120°, find the values of each of the other three angles in degrees.

If two complementary adjacent angles be bisected, what is the value of the angle formed by the bisectors in degrees? In grades?

What is the complement of $37° \, 44' \, 51''$?

What is the supplement of $104° \, 33' \, 21''$?

What is the complement of $41^g \, 28^{\backprime} \, 32^{\backprime\backprime}$?

What is the supplement of $125^g \, 76^{\backprime} \, 84^{\backprime\backprime}$?

An angle of 60^g is how many degrees?

An angle of 99° is how many grades?

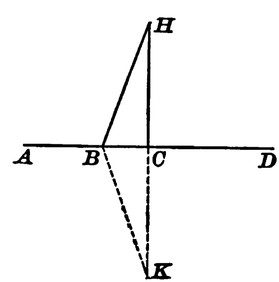
One-half a right angle is what part of three right angles? Of two right angles?

One-fifth of two right angles is what part of one right angle? Of four right angles?

How many *degrees* in an angle that is one-third its complement? How many grades?

If one of two complementary angles is acute, what kind of an angle must the other be?

If one of two supplementary angles is acute, what kind must the other one be?



71. A perpendicular is the shortest line that can be drawn from a point to a straight line.

(The analysis of this theorem has been given in the Introduction.)

Post. Let *AD* be any st. line, and *HC* a line drawn from pt. *H* ⊥ to *AD*.

Also let *HB* be *any other line* that can be drawn from pt. *H* to *AD*.

We are to prove that $HC < HB$.

Cons. Extend HC from pt. C, making $CK = CH$, and join BK.

Dem. Conceive the figure BHC to be revolved on BC as an axis until it comes into the same plane with figure BCK.

Then the line CH will fall upon CK.

(Sect. 36 and Theorem 61.)

Why ?

Pt. H will fall upon pt. K. Why ?

Line BH will coincide with line BK.

What is the relation, then, between BH and BK?

$$KC + CH < BH + BK, \qquad \text{Why ?}$$

or $$CH + CH < BH + BH. \qquad \text{How obtained ?}$$

Whence $$2\,CH < 2\,BH, \qquad \text{Why ?}$$

or $$CH < BH. \qquad \text{Why?}$$

But BH represents *any line* that can be drawn from pt. H to line AD *other than* the perpendicular CH. Hence, from our last inequality, the perpendicular is in every case the shorter. Q.E.D.

72. The *converse* of the previous theorem; viz.:

Hyp. If several lines be drawn from a point to a straight line, one of which is the shortest that can be drawn,

Con. the latter only will be perpendicular to the said straight line.

Sug. Let AB be the shortest line that can be drawn from pt. A to line CD, and from A draw a perpendicular to CD.

By Theorem 76, what must this $\perp$ be?

How must it and line AB be situated relatively, then?

Remarks. (*a*) The perpendicular distance from a point to a line is usually termed simply *the distance*.

(*b*) When a $\perp$ is drawn to another line, this other line is called the *base* of the $\perp$; and the pt. where the $\perp$ meets the base is called the *foot* of the $\perp$.

73. If two oblique lines be drawn from the same point in a perpendicular to its base, so as to cut off equal distances from the foot of the perpendicular,

I. the two oblique lines will be equal;

II. the two angles which the oblique lines make with the perpendicular are equal;

III. the two angles formed by the two oblique lines and the base are equal.

Sug. Use method similar to that in 71.

74. If a perpendicular be drawn to a line at its middle point, and any point in the perpendicular be selected at random, this point is the same distance from one extremity of the given line as from the other.

75. If a perpendicular bisect a given line, and any point outside the perpendicular be selected at random, the distances of this point from the extremities of the given line are unequal.

76. If two lines be drawn from a point to the extremities of a given line, and two other lines be drawn from a point within the first two to the same extremities, the sum of the first two lines will exceed the sum of the other two.

Sug. Extend one of the second pair until it meets one of the first pair.

77. If from the same point in a perpendicular two oblique lines be drawn to the base so as to cut off unequal distances from the foot of the perpendicular, the one cutting the base the farther from the foot will be the longer.

Sug. In constructing diagram for this *Dem.* be sure that both oblique lines are not on the same side of the $\perp$. Consult 73 and 76.

78. Converse of 73, I.

Sug. Use method similar to that illustrated in 66; i.e. designating the distances cut off by x and y, then $x > y$, or $x < y$, or $x = y$. Prove that the first two relations are impossible.

79. Converse of 77.

Sug. Use method similar to that in 78.

80. Only two equal straight lines can be drawn from the same point in a perpendicular to its base, or from a point to a straight line.

81. From a point without a straight line only one perpendicular to that line can be drawn; or if more be drawn, they must coincide.

82. If two points, each of which is equally distant from the extremities of a given line, be joined by a line, this line or its extension will bisect the given line and be perpendicular to it.

83. If two lines in the same plane be perpendicular to the same straight line, they are parallel.

Sug. They must either be parallel or meet. Show the impossibility of the latter condition.

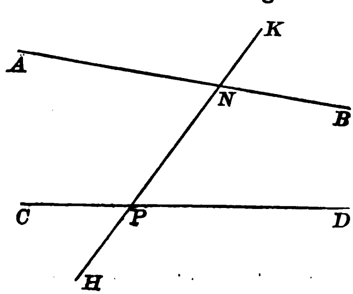
84. If a line is perpendicular to one of two parallel lines, it will be perpendicular to the other also.

85. If two straight lines be parallel to a third, they are parallel to each other.

Sug. Construct a perpendicular to one, and then consult 83.

TRANSVERSALS.

86. When two straight lines in the same plane are crossed by another, the latter is called a *transversal.*



In the figure, which line is the *transversal?*

How many angles are formed?

Certain ones of the above angles are termed *interior angles.* Name them.

Why are they called *interior angles?*

What name would you give to the others to distinguish them from those already named? Why?

How many pairs of *vertical* angles? Name them.

How many pairs of *adjacent* angles? Name them.

Two angles situated on opposite sides of the transversal, either both interior or both exterior, and not adjacent, are called *alternate* angles.

How many pairs of *alternate interior* angles? Name them.

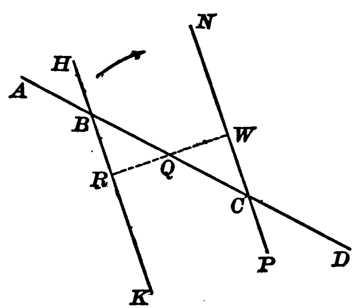
How many pairs of *alternate exterior* angles? Name them.

Two non-adjacent, non-vertical angles, of which one is interior and the other exterior, and both on the same side of transversal, are called technically *exterior interior* angles.

How many pairs of *exterior interior* angles? Name them.

87. *Hyp.* If two parallel lines be crossed by a transversal,

Con. the alternate interior angles are equal.



Post. Let, etc. (The pupil should give it *with precision.*)

We are to prove that either pair of *Alt. Int.* angles, selected at random, are equal; e.g.:

that $\angle KBC = \angle BCN$.

Cons. Through Q, the middle point of BC, construct RW perpendicular to one of the two parallels, as HK.

Dem. What is the position of RW relative to NP? Why?

What relation in regard to magnitude exists between the angles BQR and WQC? Why?

Conceive the entire figure $AHBQRK$ to revolve about Q as a pivot, keeping always in the same plane, until the line QA coincides with QD.

When QA has reached the position of QD, where must the point B be? Why?

What position must the line QR take? Why?

What must be the relative position, then, of the lines *CW* and *BR*? Why? (See 81.)

What must be the relative positions, then, of the points *R* and *W*? Why?

What is true, then, of the sides and vertices of the two angles *KBC* and *BCN*?

What must be the relation, then, in respect to magnitude, between these two angles?

Hence, etc. Q.E.D.

88. If two parallel lines be crossed by a transversal, prove

I. that the exterior interior angles are equal; (*Sug.* Consult 87 and 56.)

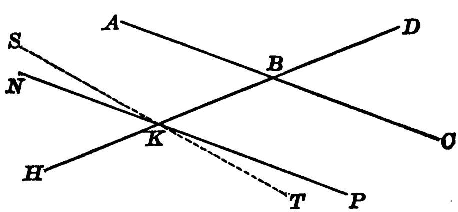
II. that the alternate exterior angles are equal;

III. the two interior angles on the same side of the transversal are supplementary;

IV. that the two exterior angles on the same side of the transversal are supplementary.

89. *Hyp.* If two lines in the same plane be crossed by a transversal so as to make the exterior interior angles equal,

Con. these two lines will be parallel.



Post. Let *AC* and *NP* be two lines in the same plane crossed by the transversal *DH*, making the angles *DBA* and *BKN* equal.

We are to prove that *AC* and *NP* are parallel.

Dem. It is evident that they must occupy one of the two relative positions; viz. *parallel* or *non-parallel.*

Let us adopt the supposition that they are *non-parallel*.

Then a line may be drawn through point K, parallel to AC. Let the dotted line ST represent this line.

Then $\angle DBA = \angle BKS$. (Theorem 88 I.)

But $\angle DBA = \angle BKN$. (By Hyp.)

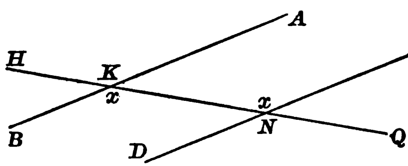
$\therefore \angle BKS = \angle BKN$. (Axiom III.)

The pupil should finish the argument.

Sug. Consult the argument in the demonstration of Theorem 66.

90. *Hyp.* If two lines in the same plane be crossed by a transversal so as to make the alternate interior angles equal,

Con. these two lines will be parallel.



Post. Let AB and CD be two lines in the same plane crossed by the transversal HQ, making $\angle CNK = \angle NKB$.

We are to prove that AB and CD are parallel.

Dem. $\angle CNK = \angle QND$. Why?

$\angle CNK = \angle NKB$. Why?

$\therefore \angle QND = \angle NKB$. Why?

$\therefore$ CD and AB are parallel. (Theorem 89.)

Q.E.D.

91. Converse of 38, II., III., and IV.

91 (a). If two lines be crossed by a transversal making

I. the alternate interior or exterior angles unequal, or

II. the exterior interior angles unequal, or

III. the two interior, or two exterior, angles, on the same side of the transversal, together less than a straight angle, the two

lines will meet if sufficiently extended, and on that side of the transversal on which the sum of the interior angles is less than a straight angle.

√ **92.** If two parallel lines be crossed by a transversal, the bisectors of the alternate interior, or alternate exterior, angles are parallel.

√ **93.** If two parallel lines be crossed by a transversal, the bisectors of the two interior, or the two exterior angles, on the same side of the transversal, are perpendicular to each other.

Sug. Consult 64, 92, and 84.

94. If two angles have their sides parallel, two and two, they are either equal or supplementary.

95. If two angles have their sides respectively perpendicular to each other, these angles are either equal or supplementary.

Which of the above results will be true if the angles are both acute? If both obtuse? If one is acute and the other obtuse? If both are right angles?

TRIANGLES.

96. A *triangle* is a plane figure bounded by three straight lines, and consequently has three angles. It is sometimes called a *trigon.*

Triangles are named,

I. with reference to the character of their angles;

II. with reference to the relations between their sides.

I. If a triangle has all its angles *acute,* it is called an *acute triangle.*

If it has one *obtuse* angle, it is called an *obtuse triangle.*

If it has one *right* angle, it is called a *right triangle.*

II. If all its sides are *equal,* it is called an *equilateral triangle.*

If two of its sides are *equal,* it is called an *isosceles triangle.*

If no two of its sides are *equal,* it is called a *scalene triangle.*

The term *isosceles* is used to designate the fact that two sides are equal without any reference to the third side, even though it may then be known, or afterwards ascertained, that the sides are all equal.

Construct triangles to illustrate each of the above varieties.

Construct *three* triangles that shall illustrate all the six varieties.

The sum of the three sides of a triangle is called the *perimeter.*

In a right triangle, the side opposite the right angle is called the *hypothenuse,* and the other two sides the *legs.*

The side on which a triangle is supposed to rest is called the *base.* In general, any side of a triangle may be considered the base; but in an isosceles triangle the *third* side, and in a right triangle one of the *legs,* is generally the base.

The angle opposite the base is called its *vertical* angle; and its vertex, the vertex of the triangle.

The perpendicular distance from vertex to base, or base extended, is called the *altitude* of a triangle.

The line drawn from the vertex of a triangle to the middle point of the base is called a *median.*

97. Any side of a triangle is less than the sum of the other two, and greater than their difference.

98. The sum of the three angles of a triangle is equal to two right angles.

Sug. Consult Theorems 58 and 87.

99. If one side of a triangle be extended, the exterior angle thus formed is equal to the sum of the two interior non-adjacent angles.

100. If two angles of a triangle be known, how can the third be found?

Two angles of a triangle are $34° 28' 42''$ and $29° 44' 56''$, respectively; find the value of the third angle.

Two angles of a triangle are $64^g\,75'\,80''$ and $86^g\,45'\,75''$, respectively; find value of the third angle.

If one angle of a triangle be a right angle, what relation exists between the other two?

How many right angles in a triangle? How many obtuse angles?

If the three angles of a triangle are all equal, what is the value of each angle in terms of a right angle? In degrees? In grades?

If two angles of one triangle be equal respectively to two angles of another, what relation exists between their third angles? Prove.

If one acute angle of a right triangle is $27°\,38'\,50''$, what is the value of the other acute angle?

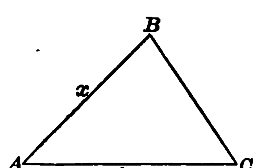
Find values of the three angles of a triangle, if the second is three times the first and the third is two times the second.

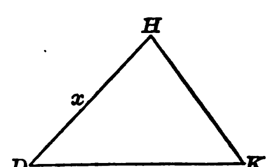
In a right triangle one of the acute angles is $17°\,40'$ greater than the other; find the values of both acute angles.

The sum of two angles is 30^g, and their difference is $9°$; find each angle.

101. *Hyp.* If two triangles have two sides and their included angle, of one, equal respectively to two sides and their included angle, of the other,

Con. the two triangles will be equal in all respects.





Post. Let ABC and DHK be two triangles having side $AB = DH$, $AC = DK$, and their included angles A and D equal.

We are to prove that the triangle ABC equals the triangle

DHK; i.e. that all their remaining corresponding parts are respectively equal.

Dem. Conceive the triangle *ABC* to be so applied to the triangle *DHK* that side *AB* shall fall upon *DH*, the point *A* falling on point *D*.

Where must the pt. *B* fall? Why?

Where must the line *AC* fall? Why? (Sect. 36.)

Where must pt. *C* fall? Why?

What, then, must be the relative position of *BC* and *HK*? Why?

Hence, the two triangles are *coincident*, and are therefore equal. (Sect. 34.)

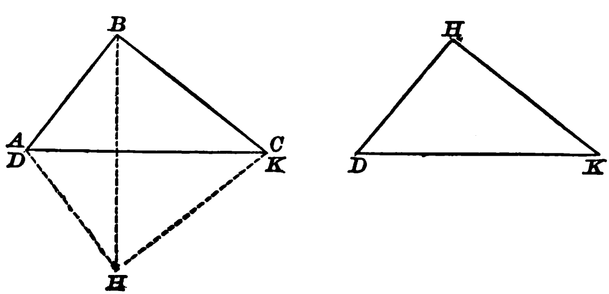
Q.E.D.

Remark. The corresponding sides and angles of two triangles are those that are similarly situated, and are called *homologous*. If the triangles are equal, the *homologous sides* are those that are opposite the equal angles; and conversely, the *homologous angles* are those that are opposite the equal sides.

102. If two triangles have two angles and the side connecting their vertices, of one, equal respectively to two angles and side connecting their vertices, of the other, the two triangles will be equal in every respect.

Sug. Use method similar to the preceding.

103. If the three sides of one triangle are equal respectively to the three sides of another, the two triangles are equal in every respect.



Sug. Apply the triangles as indicated above, then consult 82, 73, II., and 101.

104. If two right triangles have their legs respectively equal, they are equal in all respects.

105. If two right triangles have a leg and an acute angle of one equal respectively to a leg and corresponding acute angle of the other, they are equal in all respects.

106. If two right triangles have the hypothenuse and an acute angle of one equal respectively to the hypothenuse and an acute angle of the other, they are equal in all respects.

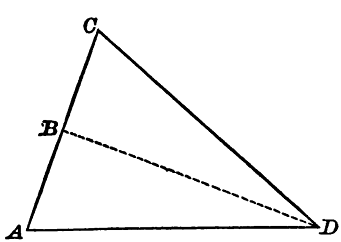
Query. Would two right triangles be equal if they had *a side* and an acute angle of one equal to *a side* and an acute angle of the other?

107. If two right triangles have the hypothenuse and a leg of one equal respectively to the hypothenuse and a leg of the other, they are equal in all respects.

Sug. Apply one triangle to the other so that the equal legs shall coincide; then consult 78 and 103.

108. *Hyp.* If a triangle have two of its sides equal,

Con. the angles opposite those sides are also equal.



Post. Let ACD be a triangle having the sides AD and CD equal.

We are to prove that

$$\angle A = \angle C.$$

Cons. Draw $DB \perp$ to AC.

Dem. The two triangles CBD and ABD are rt. $\triangle$. Why?

Name the hypothenuse of each.

What line is a leg of both triangles?

∴ The two $\triangle$s are equal. (Theorem 107.)

∴ $\angle C = \angle A$. (Being homologous angles of equal $\triangle$s.)

Q.E.D.

Query. In the above demonstration how else could the line *BD* have been drawn? Give the corresponding variation in demonstration.

The pupil should also understand now, at the outset, that only *one condition* can be imposed, in drawing an auxiliary line, which *fixes its direction.* For instance, he cannot say, in the previous construction, "Draw *DB* perpendicular to *CA* through its middle point," since either condition gives the line a definite direction; and, previous to demonstration, he does not know but that one condition will conflict with the other.

109. Converse of 108.

Sug. Employ method similar to that in 108.

110. If a triangle be equilateral, it will also be equiangular.

Sug. Consult 108.

111. Converse of 110.

Sug. Consult 109.

112. If a perpendicular bisect the base of an isosceles triangle, it will pass through the vertex and bisect the vertical angle.

113. Converse of 112.

114. If a line be drawn from the vertex of an isosceles triangle to the middle point of the base, this line will bisect the vertical angle and be perpendicular to the base.

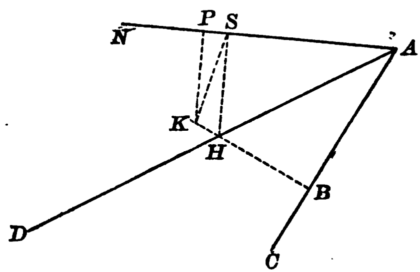
115. If one of the equal sides of an isosceles triangle be extended at the vertex, and the exterior angle thus formed be bisected, the bisecting line will be parallel to the base.

116. Converse of 115.

117. If a line bisect an angle, any point selected at random in this bisector will be equally distant from the sides of the angle.

Sug. Consult 72, Remark (*a*), and 106.

118. If a line bisect an angle, any point selected at random outside this bisector is unequally distant from the sides of the angle.



Sug. If K be the point, then what is the relation between HS and HB? Why?

What is the relation between BK and $KH + HS$? Why?

What is the relation of KS to $KH + HS$? Why?

What is its relation, then, to BK?

What is the relation between KS and KP?

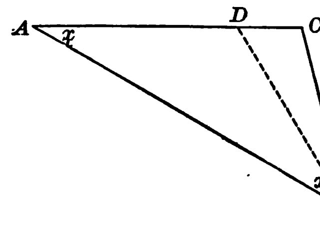
What, then, must be the relation between KP and BK?

119. If a point be equally distant from the sides of an angle, the line joining it with the vertex will bisect the angle.

120. If the angles at the base of an isosceles triangle be bisected, the bisectors will form, with the base, an isosceles triangle.

121. If the angles at the base of an isosceles triangle be double the vertical angle, a line bisecting either of the former will divide the triangle into two isosceles triangles.

122. If two angles of a triangle be unequal, the side opposite the greater of these two angles is longer than the side opposite the lesser.



Post. Let ABC be any triangle with $\angle B > \angle A$.

We are to prove that the side AC, opposite the greater angle B, is longer than the side CB, opposite the lesser angle A.

Cons. Let BD be drawn so as to cut off a portion of the larger angle B, making it equal to angle A, so that they will both be in the same triangle as DAB.

What relation between AD and DB? Why?

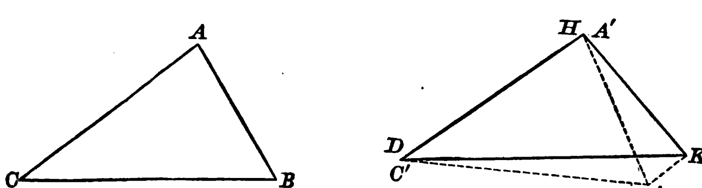
What relation between BC and $CD + DB$? Why?

(The pupil should finish it without difficulty.)

123. Converse of 122.

Sug. Three possible relations between the two angles. Prove the impossibility of two of them; or, a method similar to that in 122 may be employed.

123 (*a*). If two triangles have the two sides of one equal respectively to two sides of the other, but the included angles unequal, then the third side of that triangle having the greater included angle is longer than the third side of the other.



Sug. Apply the two triangles so that two of the equal sides shall coincide, as is represented in the above diagram.

Will the other pair of equal sides coincide? Why?

Join the other two vertices.

What kind of a triangle is *HKB'*?

What relation, then, between angles *HKB'* and *HB'K*?

What must be the relation, therefore, between the angles *DKB'* and *C'B'K*?

Consult Theorem 122.

Or, instead of joining *B'* and *K*, the angle *KA'B'* may be bisected, and the point where this bisector cuts *DK* joined with B'. Then consult 101 and 97.

124. Converse of 123.

Sug. Therė are only *three* possible relations between those angles. Prove the impossibility of two of them.

Advance Theorems.

125. The bisectors of the three angles of a triangle meet in one point.

Sug. From the point of intersection of *two* of the bisectors draw a line to the other vertex, and also perpendiculars to the three sides. Prove the former a bisector by means of equality of triangles.

126. The perpendiculars which bisect the three sides of a triangle meet in a point.

Sug. From the point of intersection of two of the perpendiculars draw a line to the middle point of the other side. Prove this line perpendicular, by consulting 74 and 103.

127. The perpendiculars from the three vertices of a triangle to the opposite sides meet in one point.

Sug. Through each vertex draw a line parallel to the opposite side. Then consult 102 and 127.

128. If two angles of an equilateral triangle be bisected, and lines be drawn through the point of intersection parallel to the sides, the sides will be trisected.

QUADRILATERALS.

129. If two lines in the same plane be crossed by two transversals, a figure of four sides may be formed, which is called a *quadrilateral.* Hence a *quadrilateral* may be defined as a plane figure bounded by four straight lines. A quadrilateral is also called a *tetragon.*

If each of the two pairs of lines is *parallel,* the quadrilateral thus formed is called a *parallelogram.*

Define, then, a parallelogram.

If a parallelogram have all its sides *equal,* it is called a *rhombus.*

If its angles are right angles, it is called a *rectangle.*

If it is both a rhombus and a rectangle, it is called a *square.*

Give all the names applicable to a square.

If a quadrilateral have only two of its sides parallel, it is called a *trapezoid.*

If no two of its sides are parallel, it is called a *trapezium.*

A parallelogram whose angles are oblique and adjacent sides unequal is sometimes called a *rhomboid.*

A rectangle whose adjacent sides are unequal is sometimes called an *oblong.*

The line which joins two opposite vertices of a quadrilateral is called its *diagonal.*

The side upon which a parallelogram is conceived to rest, and the side opposite the latter, are termed, respectively, the *lower* and *upper bases.*

The parallel sides of a trapezoid are always considered as its bases, the other two sides its *legs,* while the line bisecting the legs is called the *median.*

The *altitude* of either a parallelogram or trapezoid is the perpendicular distance between its bases.

The pupil should construct figures to represent all the above-mentioned quantities.

The sum of the four sides of a quadrilateral is called its *perimeter.*

If the *legs* of a trapezoid are equal, it is called an *isosceles trapezoid.*

130. Either diagonal divides the parallelogram into two equal triangles.

Sug. Consult 87 and 102.

Query. Is the converse of this theorem true?

131. The sum of the four angles of a quadrilateral equals four right angles.

132. The opposite sides of a parallelogram are equal.

133. Converse of 132.

Sug. Draw one diagonal, then consult 45 and 34.

134. The opposite angles of a parallelogram are equal.

135. Converse of 134.

Sug. Consult 131 and 91.

136. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

137. The two diagonals of a parallelogram bisect each other.

138. Converse of 137.

139. If one angle of a parallelogram be a right angle, the other three angles are right angles, and the parallelogram is therefore a rectangle.

140. The diagonals of a rectangle are equal.

141. Converse of 140.

142. The diagonals of a rhombus are perpendicular to each other.

143. Converse of 142.

144. The diagonals of a rhombus bisect the angles.

145. Converse of 144.

146. If two parallel lines be crossed by a transversal, the bisectors of the interior angles form a rectangle.

Sug. Consult 88, III., 92, 93, and 64.

147. The lines which bisect the angles of a rhomboid form a rectangle.

148. If two parallelograms have two sides and their included angle of one equal respectively to two sides and their included angle of the other, the two parallelograms are equal in every respect.

149. If a line be drawn parallel to one side of a triangle, and bisecting another side, this line will bisect the third side of the triangle, and be equal to one-half the side parallel to it.

150. Converse of 149.

151. The median of a trapezoid is parallel to its bases and equal to one-half their sum.

Sug. Through one extremity of the median draw a line parallel to one leg, and extend the shorter base to meet it.

Advance Theorems.

152. If from any point in the base of an isosceles triangle lines are drawn parallel to the equal sides, the perimeter of the parallelogram thus formed will be equal to the sum of the two equal sides of the triangle.

153. If the legs of a trapezoid are equal, the angles which they make with either base are equal.

154. Converse of 153.

155. If the angles at one base of a trapezoid are equal, the angles at the other base are also equal.

156. The line which is parallel to the bases of a trapezoid and bisects one leg is a median.

157. The line joining the vertex of the right angle to the middle point of the hypothenuse in a right triangle is equal to one-half the hypothenuse.

158. The lines which join the middle points of the sides of a triangle divide the triangle into four equal triangles.

159. The three medians of a triangle meet in one point.

Sug. Through the point of intersection of the two medians draw a line from the third vertex. Join the middle points of those segments of the two medians between the vertices and their point of intersection. Join also the extremities of the two medians. Thus a parallelogram may be formed.

160. The lines which join the middle points of the sides of any quadrilateral form a parallelogram.

Sug. Draw the diagonals, then consult 150.

161. The lines which join the middle points of the sides of a rhombus form a rectangle.

162. The lines which join the middle points of the sides of a square form a square.

163. The lines which join the middle points of the sides of a rectangle form a rhombus.

164. The lines which join the middle points of the sides of an isosceles trapezoid form a rhombus.

165. The median of a trapezoid bisects both diagonals.

Sug. Consult 151 and 149.

166. The diagonals of an isosceles trapezoid are equal.

167. Converse of 166.

168. If a trapezoid be isosceles, the opposite angles are supplementary.

169. The line which joins the middle points of the diagonals of a trapezoid equals one-half the difference of the bases.

170. The two perpendiculars from the extremities of the base to the equal sides of an isosceles triangle are equal.

171. The medians drawn to the equal sides of an isosceles triangle are equal.

172. The bisectors of the angles at the base of an isosceles triangle are equal.

173. The two perpendiculars from the middle point of the base of an isosceles triangle to the equal sides are equal.

174. State and prove the converse of each of 170, 171, 172, 173.

175. If one of the equal sides of an isosceles triangle be extended at the vertex, making the extension equal to the side, the line joining the end of the extension with the nearer extremity of the base is perpendicular to the base.

176. If one angle of an isosceles triangle be 60°, the triangle is equilateral.

177. If from the middle points of two opposite sides of a parallelogram lines be drawn to the vertices of the angles opposite, these lines will trisect the diagonal that joins the other two vertices.

178. If the two base angles of a triangle are bisected, and through the point of intersection of these bisectors a line be drawn parallel to the base and terminating in the sides, this line is equal to the sum of the two segments of the sides between this parallel and the base.

179. The bisectors of the vertical angle of a triangle and the angles formed by extending the sides below the base meet in a point which is equally distant from the base and the extensions of the sides.

180. If one of the acute angles of a right triangle is double the other, the hypothenuse is double the shorter leg.

181. If from any two points selected at random in the base of an isosceles triangle perpendiculars be drawn to the equal sides, the sum of the perpendiculars from one point equals the sum of the perpendiculars from the other point.

182. If from any point selected at random in an equilateral triangle perpendiculars be drawn to the sides, the sum of these perpendiculars is constant, and equal to the altitude of the triangle.

CIRCLES.

183. A *circle* is a portion of a plane bounded by a curved line, all points of which are equally distant from a point within called the *centre.*

The bounding line is called the *circumference.*

Any portion of the circumference is called an *arc.*

A *radius* (plural *radii*) is any line from centre to circumference.

A *diameter* is any line passing through the centre and terminating ~~both ways~~ in the circumference.

How do the radius and diameter of the same circle compare in magnitude?

How do the radii of a circle compare in magnitude? The diameters?

A *semicircumference* is one-half the circumference.

A *sector* is that part of a circle included between an arc and the radii drawn to its extremities.

A *quadrant* is a sector which is one-fourth the circle.

A *quadrant arc* is one-fourth the circumference.

A *chord* is any straight line whose extremities are in the circumference.

A *segment* is that portion of the circle included between an arc and the chord which joins its extremities.

Every chord, therefore, must divide the circumference into two arcs and the circle into two segments.

If the arcs are unequal, they are designated as *major* and *minor* arcs, and the segments as *major* and *minor* segments.

The chord is said to *subtend* the arc, and the arc is said to be *subtended* by the chord.

Whenever a chord and its subtended arc are mentioned, the *minor arc* is meant unless it is otherwise specified.

If two circles have the same centre, they are said to be *concentric.*

A *central angle* is an angle formed by two radii.

An *inscribed angle* is an angle formed by two chords, with its vertex in the circumference.

When would an angle be said to be inscribed in a segment?

A *tangent* is a straight line that *touches* the circumference of a circle, but on being extended does not intersect it; i.e. the tangent and the circumference have *one point,* and only one, in common. This point is called the *point of contact,* or point of tangency.

Two circumferences are *tangent to each other* when they have one point in common but do not intersect; i.e. when they *touch* each other.

If one of two tangent circumferences lies within the other, they are said to be tangent *internally;* if it lies without, they are said to be tangent *externally.*

A *secant* is a straight line that intersects a circumference in two points lying partly within and partly without the circle; e.g. a chord extended in either direction becomes a secant.

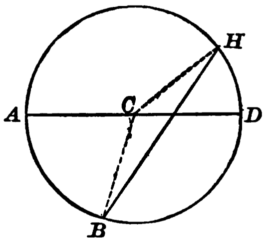
The term *circle* is also sometimes used to designate a circumference.

Construct diagrams illustrating the above magnitudes and their relations.

184. A diameter is greater than any other chord.

Sug. Draw the diameter *AD*, and let *HB* be any other chord. Join *CH* and *CB*. Consult 97.

185. A diameter bisects both the circle and circumference.



Sug. Fold over one part on the other, using the diameter as an axis of revolution.

185 (*a*). Converse of 185.

186. A straight line cannot intersect the circumference of a circle in more than two points.

Sug. Suppose it could intersect in *three* points, and draw radii to the three points; then consult 80.

187. If two circles have equal radii, they are equal.

Sug. Apply one to the other.

188. If the diameters of two circles are equal, the circles are equal.

189. Converse of 187 and 188.

190. If two equal circles be concentric, their circumferences will coincide.

191. If, in the same or equal circles, two central angles be equal, the arcs which their sides intercept will also be equal.

Remark. In this and the following theorems where the expression "same or equal circles" occurs, the demonstration will be more satisfactory if two circles are used.

Sug. Apply one circle to the other.

192. Converse of 191.

193. If, in the same or equal circles, two chords be equal, the arcs which those chords subtend are also equal.

Sug. Draw radii to the extremities of the chords, then consult 103 and 191.

194. Converse of 193.

Sug. Draw radii as above. Consult 192 and 101.

195. If, in the same or equal circles, two central angles be unequal, the arc intercepted by the sides of the greater angle is greater than the arc intercepted by the sides of the lesser angle.

Sug. Apply one circle to the other.

196. Converse of 195.

197. If, in the same or equal circles, two chords be unequal, the arc subtended by the greater chord is greater than that subtended by the lesser.

Sug. Draw radii to extremities, then consult 124 and 195.

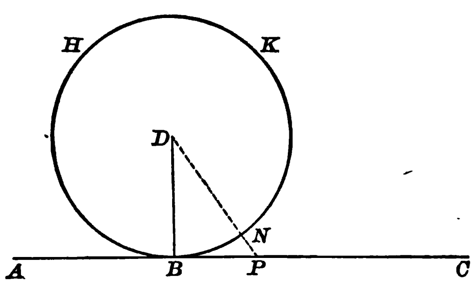
198. Converse of 197.

Sug. Draw radii as above, then consult 196 and 123. Or, three possible relations.

199. If a diameter (or radius) be perpendicular to a chord, it will bisect the chord, and also the arcs into which the chord divides the circumference.

Sug. Draw radii to extremities of the chord, then consult 78 or 107 and 191.

200. A straight line that is perpendicular to a radius at its extremity is a tangent to the circle.



Post. Let BHK be a circle, DB a radius, and AC a straight line perpendicular to DB and passing through the point B.

We are to prove that AC is a tangent to the circle BHK.

Now, if we can prove that *every point in AC except B is outside the circumference*, then AC must be a tangent. (See Def. of Tangent.)

Cons. From D draw *any other* line to AC, as DP.

Then P, the extremity of DP, must represent *any point in AC except B.*

Now $DP > DB$. Why?

But DB is a radius, and if DP is longer than a radius, where must its extremity be?

Hence, etc.

201. If a radius be drawn to the point of contact of a tangent to a circle, it will be perpendicular to the tangent.

Sug. If it can be proved that the radius is the *shortest distance* from the centre to the tangent, then it must be perpendicular. (See Theorem 72.) Use previous diagram.

What must be the situation of all the points in the tangent, except the point of contact, with reference to the circumference of the circle?

What, then, must be the relation to the radius of a line drawn from the centre to any point in the tangent except the point of contact?

Hence, etc.

202. If a perpendicular be erected to a tangent at the point of contact, this perpendicular, if extended, will pass through the centre of the circle.

Sug. Draw a radius to point of contact, then consult 201 and 65.

203. If a line be drawn from the centre of a circle perpendicular to a tangent, this line will pass through the point of contact.

204. If a chord and tangent be parallel, the arcs which they intercept are equal.

Sug. Draw a radius to point of contact, then consult 201, 84, and 199.

205. If two chords be parallel, the arcs which they intercept are equal.

206. If two tangents be parallel, the arcs which they intercept are equal.

207. The line joining the points of contact of two parallel tangents is a diameter.

208. If two tangents have their points of contact the opposite extremities of a diameter, the tangents are parallel.

209. If a radius bisect a chord, it will bisect the arc and be perpendicular to the chord.

210. If a perpendicular bisect a chord, it will, if extended, pass through the centre of the circle.

211. If a radius bisect an arc, it will also bisect its subtending chord and be perpendicular to it.

211 (a). If a line bisect an arc and its subtending chord, this line will, if extended, pass through the centre of the circle.

212. If two non-intersecting chords intercept equal arcs, they are parallel.

213. Converse of 204.

214. If, in the same or equal circles, two chords be equal, they are at equal distances from the centres.

Sug. Consult 72, Remark (a).

215. Converse of 214.

216. If, in the same or equal circles, two chords be unequal, the less chord is the farther from the centre.

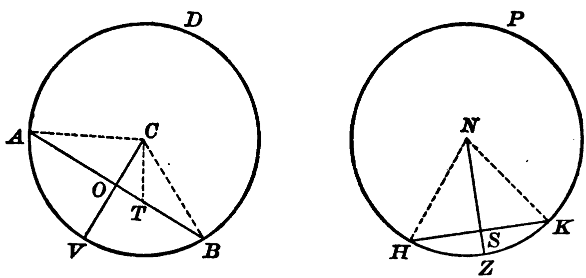
Post. Let ABD and HPK be two equal circles, and chord $AB >$ chord HK.

We are to prove that HK is farther from the centre N than AB is from the centre C.

Cons. Draw the radius CV perpendicular to AB and radius NZ perpendicular to HK. Draw also the radii CA, CB, NH, and NK.

Then CO is the distance AB is from C, and NS is the distance HK is from N. (72, Remark (*a*).)

So we are, in reality, to prove $NS > CO$.



Dem. $AB > HK$. Why?

OB is what part of AB? Why?

HS is what part of HK? Why?

Then what is the relation of OB and HS in respect to magnitude?

Express that relation by symbols.

What is the relation between the two arcs AVB and HZK? Why?

What relation, then, between the $\angle$s ACB and HNK? Why?

How, then, does the sum of the angles A and B compare with the sum of the angles H and N? Explain.

How does $\angle A$ compare with $\angle B$? $\angle H$ with $\angle K$? Why?

What relation, then, in respect to magnitude, between the $\angle$s H and B? Why?

Now, on BO mark off a distance from B equal to HS, as BT, and join CT.

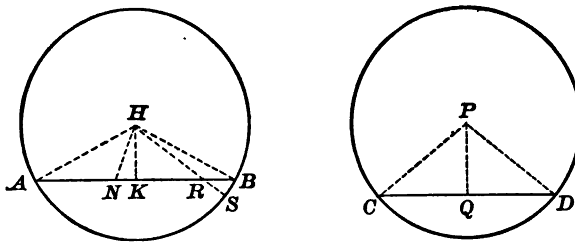
Compare CT and NS. (See Theorem 123 (*a*).)

Now compare them both with CO.

Hence, etc.

The following variation of the above method was given by one of the author's pupils.

Make $AR = CD$. Then, since HR is less than PD, $\angle A$ is less than $\angle C$. (Theorem 124.)



The rest is similar to the one above.

217. Converse of 216.

Sug. Three possible relations.

218. Through any three points selected at random, provided, however, they are not in the same straight line, one circumference can be made to pass, and only one; or, if more be drawn, they must coincide.

Sug. If the points be joined, these lines must be chords of the circle. Then consult 210.

219. If two unequal circles are concentric, the chords of the greater which are tangents of the less are equal.

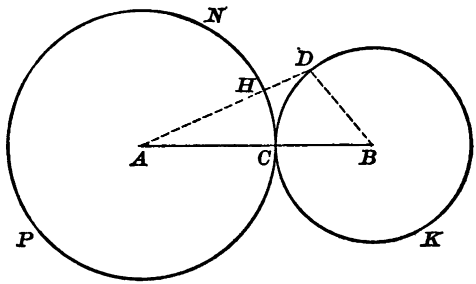
Sug. Consult 201, 215, and 72, Remark (a).

220. If two circles be tangent to each other externally, the radii drawn to the point of contact form one and the same straight line.

Post. Let the two circles PCN and CDK be tangent to each other externally at C, and AC and BC the radii drawn to the point of contact C.

We are to prove that ACB is a straight line. If we can prove that ACB is the *shortest distance* from A to B, then ACB must be a straight line. For if a straight line be drawn

from A to B, it will be the ***shortest distance*** between A and B. (Sect. 42, XIV.) And since there can be but ***one shortest distance***, if ACB is proved also to be the shortest distance, then ACB must coincide with that line, and consequently be a straight line.



So we are to prove that ACB is the ***shortest distance*** from A to B.

Cons. Draw any other radius than BC in circle CDK, as BD, and join AD.

Dem. Since every point in the circumference CDK except C is outside the circumference CPN (see Def.), AD must extend beyond the circumference, and

$$\therefore \quad AD > AC. \qquad \text{Why?}$$

$$DB = CB. \qquad \text{Why?}$$

$$\therefore \quad AD + DB > AC + CB. \qquad \text{Why?}$$

But D is any point in the circumference CDK except C. Hence, the distance from A to B by way of D is always greater than by way of C. Or ACB is the shortest distance between A and B, and is therefore a straight line. Q.E.D.

221. If radii be drawn to the point of contact of two circles that are tangent to each other externally, and a straight line be drawn through the point of contact perpendicular to one of the radii, this line will be a common tangent to the two circles.

222. If two circles be tangent to each other externally, the straight line joining their centres will pass through the point of contact.

223. If two circles be tangent to each other internally, and a line be drawn through the point of contact tangent to the outer circle, it will also be tangent to the inner circle.

224. If two circles be tangent to each other internally, and radii be drawn to point of contact, these radii will lie in the same straight line.

225. If two circles be tangent to each other internally, the line joining their centres will, if extended, pass through the point of contact.

226. If two circles be tangent to each other internally, and a common tangent be drawn through the point of contact, a perpendicular to this tangent at the point of contact will, if extended, pass through the centres of both circles.

227. If two circles be tangent to each other internally, the radius of the smaller to the point of contact will, if extended, pass through the centre of the larger, and the radius of the larger to point of contact will also pass through the centre of the smaller.

228. If two circles be tangent to each other internally, and a line be drawn from the centre of either circle perpendicular to their common tangent, this perpendicular will pass through the centre of the other circle and also through the point of contact.

229. Two unequal circles may have five positions relative to each other, viz.:

I. One may lie wholly within the other without contact.

II. They may be tangent to each other internally.

III. They may intersect each other.

IV. They may be tangent to each other externally.

V. One may lie wholly without the other without contact.

If the circles are equal, how many positions relative to each other may they have?

In Case I. what relation exists between the line joining their centres and the sum of their radii? Between the same line and the difference of their radii? Prove.

Prove both the above relations in each of the other four cases.

What must be the position of two circles, then, if the distance between their centres is

I. 0?

II. Less than the sum, and also less than the difference, of their radii?

III. Equal to the difference of their radii?

IV. Less than the sum, and greater than the difference, of their radii?

V. Equal to the sum of their radii?

VI. Greater than the sum of their radii?

230. If two circles intersect each other, the line joining their centres will bisect their common chord and be perpendicular to it.

231. If two circles intersect each other, and radii be drawn to the middle point of their common chord, these radii will form one and the same straight line.

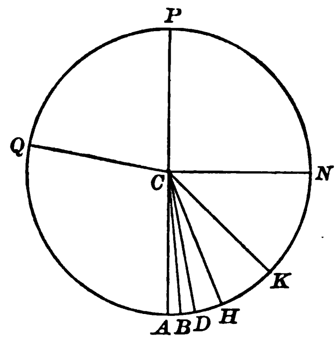
232. If two circles intersect each other, and either radius be drawn bisecting their common chord, this radius will, if extended, pass through the centre of the other circle.

233. If two circles intersect each other, and a perpendicular bisect their common chord, this perpendicular will, if extended, pass through the centres of both circles.

234.* A central angle is measured by the arc which its sides intercept on the circumference.

* See Appendix.

Dem. In the circle ANQ let us conceive the radius CA to move about C as a pivot, remaining always in the same plane. In one complete revolution it is evident that the extremity A will have traced the entire circumference. Likewise, in one-half a revolution, as when it has reached the position CP, making PCA a diameter, the extremity A will have traced one-half the circumference. So when it has reached the position CN perpendicular to AP, it will have traced one-fourth the circumference (see 199, and 59 and 60). In like manner, when it has reached the position CK, making the angles ACK and KCN equal, then the arcs AK and KN are also equal (see 191); that is, the arc AK is one-eighth of the circumference, and the angle ACK is one-eighth of the angular space about the point C. In like manner, the arc AH is one-sixteenth, AD one thirty-second, AB one sixty-fourth, etc., of the entire circumference, while their corresponding central angles are the same parts of four right angles, or the angular space about the centre C. So that, whatever the position of the radius CA, the arc described will be the same part of the entire circumference that its corresponding central angle is of the angular space about the centre. If, now, the entire circumference be divided into 360 equal parts, each part would correspond to a central angle of 1°. Consequently, if an arc contain 27 of the 360 equal parts, it may be called an arc of 27°, and its corresponding central angle would contain $\frac{27}{360}$ of four right angles, or would be an angle of 27°. Likewise, if the arc contains a whole number of degrees and a fraction, the latter can be expressed in the smaller units. For example, suppose the arc



to contain 48° 37′ 24.92″; then the corresponding central angle would contain the same number of *angular* units; i.e. it would be an angle of 48° 37′ 24.92″, to the same degree of approximation. Therefore the arc is said to *measure* the corresponding central angle, *because*, as above stated, *it is the same part of the entire circumference that its corresponding central angle is of the angular space about the centre.* Q.E.D.

235. An inscribed angle is measured by one-half the arc intercepted by its sides.

Remark. In demonstrating this theorem it will simplify it somewhat to make three cases of it; viz.:

I. When one of the sides of the angle is a diameter.

II. When the centre is between the sides of the angle.

III. When the centre is without the angle.

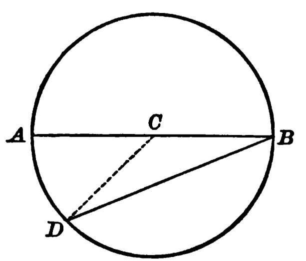


Fig. I.

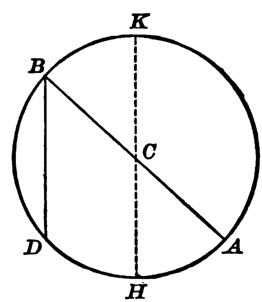


Fig. II.

Case I.—*Post.* Let B be an inscribed angle with one of its sides, as AB, a diameter.

We are to prove that the angle B is measured by one-half the arc AD; that is, one-half the arc AD is the same part of the entire circumference ABD that the angle B is of the angular space about B.

Cons. Draw the radius CD (Fig. I.).

Dem. What relation exists between CD and CB? Why?

What relation, then, between angles B and D? Why?

What relation exists between the sum of the angles B and D and angle DCA? Why?

What relation, then, exists between the angle B and the angle DCA? Why?

What measures the angle DCA? Why? (Consult 234.)

What, then, must measure the angle B?

Again, Fig. II., Case I.

Cons. Draw a diameter, as HK, parallel to BD.

Dem. What relation exists between the angles BCK and HCA? Why?

What relation, then, exists between the two arcs HA and BK? Why? (See 191.)

What relation between the two arcs DH and BK? Why?

What relation, then, between the two arcs DH and AH? Why?

What relation, then, between the two arcs AH and AD? Why?

What relation between the two angles HCA and B? Why?

What measures the angle HCA? Why?

What, then, must measure the angle B?

After answering correctly the foregoing questions the pupil should have no difficulty in writing out a complete demonstration of each of the three cases, using Case I. in demonstrating II. and III.

236. If two angles be inscribed in the same or equal circles, and their sides intercept the same or equal arcs, the two angles are equal.

236 (a). Converse of 236.

236 (b). What part of the circumference measures a right angle? Prove.

What relation exists between angles inscribed in the same segment? Why?

What measures an angle inscribed in a semicircle?

What kind of an angle, then, must it be?

If an angle be inscribed in a segment less than a semicircle, what kind of an angle must it be? Why?

If an angle be inscribed in a segment greater than a semicircle, what kind of an angle must it be? Why?

237. If an angle be formed by two chords whose vertex is between the centre and circumference, it will be measured by one-half the sum of its two intercepted arcs.

238. If the vertex of an angle formed by two secants is without the circle, this angle is measured by one-half the difference of the two intercepted arcs.

239. If an angle be formed by a tangent and chord, this angle is measured by one-half the arc intercepted by its sides.

240. If the vertex of an angle formed by a tangent and secant is without the circle, this angle is measured by one-half the difference of the two intercepted arcs.

241. If an angle be formed by two tangents to the same circle, this angle is measured by one-half the difference of the two intercepted arcs.

242. If from the same point without a circle two tangents be drawn, these two tangents are equal; i.e. the distances from the common point to the points of contact are equal.

Sug. Consult either 201 and 107, or 239 and 109.

Advance Theorems.

243. The line which joins the vertex of an angle formed by two tangents to the centre of the circle bisects the angle, and also the chord which joins the points of contact.

244. If, from the same point, two tangents to a circle be drawn whose points of contact are the extremities of a chord, the angle formed by the two tangents is double the angle formed by the chord and diameter drawn from either extremity of the chord.

245. If two circles which are tangent to each other externally have three common tangents, the one that passes through the point of contact of the two circles bisects the other two.

Sug. Consult 242.

246. If four tangents form a quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.

247. If two mutually equiangular triangles be inscribed in equal circles, the triangles are equal in all respects.

248. If two opposite sides of an inscribed quadrilateral are equal, the other two sides are parallel.

248 (*a*). Converse of 248.

249. The opposite angles of an inscribed quadrilateral are supplementary.

250. If two equal chords be drawn from opposite extremities of a diameter and on opposite sides of it, they will be parallel.

251. If two chords be drawn through the same point in a diameter making equal angles with it, they are equal.

252. If one of the equal sides of an isosceles triangle be the diameter of a circle, the circumference will bisect the base.

253. If an equilateral triangle be inscribed in a circle, and a diameter be drawn from one vertex, the triangle, formed by joining the other extremity of the diameter and the centre of the circle with one of the other vertices of the inscribed triangle, will also be equilateral.

254. The bisectors of the angles formed by extending the sides of an inscribed quadrilateral are perpendicular to each other.

255. If an equilateral triangle be inscribed in a circle, and any point in the circumference be selected at random, one of the lines which join this point to the three vertices will equal the sum of the other two.

256. If an inscribed isosceles triangle have its base angles each double the vertical angle, and its vertices be the points of contact of three tangents, these tangents will form an isosceles triangle each of whose base angles is one-third its vertical angle.

257. If through any point selected at random in a circle a diameter and other chords be drawn, the least chord will be the one that is perpendicular to the diameter.

258. ADB is a semicircle of which the centre is C, and AEC is another semicircle on the diameter AC, and AT is a common tangent to the two semicircles at A. Prove that, if from any point F in the circumference of the first a straight line be drawn to C, the part FK, cut off by the second semicircle, is equal to FH, perpendicular to the tangent AT.

259. If a triangle ABC be formed by the intersection of three tangents to a circumference whose centre is O, two of which, AM and AN, are fixed, while the third, BC, touches the circumference at a variable point P, prove

I. that the perimeter of the triangle is constant.

II. that the angle BOC is constant.

260. If the sides of any quadrilateral be the diameters of circles, the common chord of any adjacent two is parallel to the common chord of the other two.

RATIO AND PROPORTION.

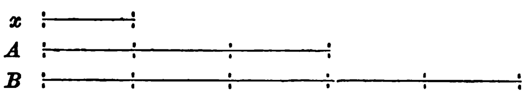
(It is assumed that the pupil is familiar with the algebraic processes involved in the manipulation of equations of the first and second degree. For definition of *Geometrical Magnitudes*, see 37.)

261. If of two unequal magnitudes the less is contained an exact number of times in the greater, the latter is said to be a *multiple* of the former, and the former an *aliquot part* of the latter.

262. If of three unequal magnitudes the smallest is contained an exact number of times in each of the two larger, the latter are said to be *commensurable*, and the former is said to be their *common measure.*

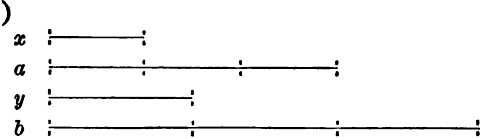
263. When no magnitude can be found which is contained an exact number of times in each of two magnitudes, the latter are said to be *incommensurable.*

264. If two commensurable magnitudes contain their common measure, one n times and the other m times, they are said to be to each other as n to m, or in the *ratio* of n to m.



For example, if the line x is contained in the line A 3 times, and in line B 5 times, then A is to B as 3 to 5, and the line x is a common measure of the two lines A and B.

265. *Equimultiples* of two or more magnitudes are the results obtained by multiplying these magnitudes by the same number. (See 39.)



Thus, if x and y be any two lines, and a and b two other lines such that the former contains the line x n times (in this particular case n is 3), and the latter contains the line y n times, then a and b are *equimultiples* of x and y respectively.

266. If two magnitudes are to each other as m to n (see 264), it is usually expressed thus, $m:n$, called the *ratio* form; or thus, $\frac{m}{n}$, called the *fractional form.*

267. Hence a *ratio* may be defined as an expression of comparison, in respect to size, of two magnitudes of the same kind.

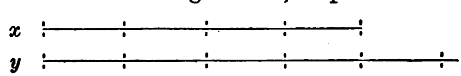
268. If x and y are two magnitudes, and y a multiple of x

(see 261), the latter being contained in the former n times, then the ratio of y to x is as n to *unity*, or $n:1$.

269. If x and y are commensurable (262), and their common measure is contained n times in x, and m times in y, then the ratio of x to y is as n to m, or, technically expressed, $n:m$.

270. If x and y are *incommensurable* (see 263), the ratio cannot be exactly expressed. If, however, x and y are numbers, the ratio may be *approximately* expressed by placing 1 for the first term, and the quotient of the greater divided by the less to any number of decimal places for the second term; thus, $1:n\,.\,pqrt+$. In such a case the computation can be carried to any required degree of approximation.

271. In a similar way can be obtained an approximate ratio of two incommensurable magnitudes, expressed numerically; e.g.:



If x and y be two incommensurable magnitudes, suppose x to be divided into a certain number, n, of equal parts, and that, on applying one of these equal parts to y, it is found to be contained m times with a remainder less than this part. Then it is evident that the ratio of y to x is neither $m:n$, nor $m+1:n$, but is greater than the former and less than the latter; i.e. greater than the value of $\frac{m}{n}$, and less than the value of $\frac{m+1}{n}$, or $\frac{m}{n}+\frac{1}{n}$. It is thus seen that the smaller the unit into which x is divided, the greater becomes the value of n, and consequently the smaller becomes the value of $\frac{1}{n}$. And hence, neglecting this value, $\frac{m}{n}$ will express the approxi-

mate ratio of y to x, the degree of approximation depending upon the size of the unit of measure.

272. The first term of a ratio is called the *antecedent,* and the second term the *consequent.*

273. A *proportion* is an expression of equality between two equal ratios, usually indicated by four dots between the two ratios; thus, $a : b :: x : y$, and read, a is to b as x is to y.

274. The four magnitudes forming a proportion are called *proportionals.*

275. If a proportion contains only two ratios, it is called a *simple* proportion; if more than two and all equal, it is called a *continued* proportion.

276. The first and last terms of a simple proportion are called the *extremes,* and the other two the *means.*

277. In a simple proportion, where the terms are all of different values, each term is said to be a *fourth proportional* to the other three.

278. In a simple proportion, where the two means or the two extremes are alike, the repeated quantity is said to be a *mean proportional* to the other two, and the proportion is called a *mean proportion.*

279. In a mean proportion, either of the quantities not repeated is said to be a *third proportional* to the other two.

280. When a line is so divided that the larger part is a mean proportional between the whole line and the smaller part, it is said to be divided in *extreme and mean ratio.*

281. It will be found on investigation that the *order* of four proportionals, *when all of the same kind,* can be varied, as well as certain other transformations effected, without destroying the proportion. The principal of these are as follows:

282. Magnitudes are said to be in proportion by *alternation* when either the two means or the two extremes are made to exchange places.

283. Magnitudes are said to be in proportion by *inversion* when the means are made to exchange places with the extremes.

284. Magnitudes are said to be in proportion by *composition* when the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of that ratio.

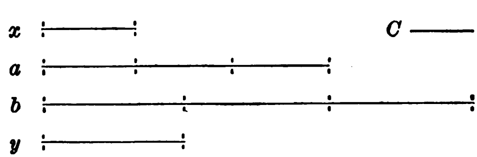
285. Magnitudes are said to be in proportion by *division* when the difference of the terms of the first ratio is to either term of that ratio as the like difference of the terms of the second ratio is to the corresponding term of that ratio.

286. Magnitudes are said to be in proportion by *composition and division* when the sum of the terms of the first ratio is to their difference as the sum of the terms of the last ratio is to their like difference.

Theorems.

287. Equimultiples of two magnitudes are in the same ratio as the magnitudes themselves.

Case I. — When the magnitudes are *commensurable*.



Let x and y be two commensurable magnitudes, and a and b be their respective equimultiples.

We are to prove $x : y : a : b$.

Dem. Since x and y are commensurable (262), they must have a common measure, as the line c. Suppose it to be contained in x n times, and in y m times; i.e. that on dividing

the two lines x and y into n and m equal parts respectively, the parts will all individually be equal to the line c. Consequently if the line c be multiplied first by n and then by m (see 39), the resulting lines will be exactly equal in length to the lines x and y respectively.

These relations may be symbolically expressed as follows:

$$(1) \qquad \frac{x}{n} = c \quad \text{and} \quad \frac{y}{m} = c.$$

$$(2) \qquad n \times c = x \quad \text{and} \quad m \times c = y.$$

Whence $$\frac{n \times c}{m \times c} = \frac{x}{y}, \quad \text{or} \quad \frac{n}{m} = \frac{x}{y}.$$

Let q be the number by which x and y are multiplied to obtain the equimultiples a and b; then

$$(3) \qquad q \times x = a \quad \text{and} \quad q \times y = b.$$

But from (2) $$q \times x = q \times n \times c,$$

and $$q \times y = q \times m \times c.$$

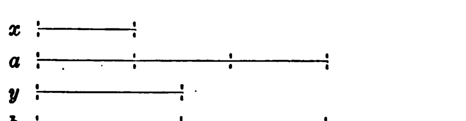
Whence $$a = q \times n \times c,$$

and $$b = q \times m \times c.$$

Therefore $$\frac{a}{b} = \frac{q \times n \times c}{q \times m \times c} = \frac{n}{m}.$$

Whence $$\frac{a}{b} = \frac{x}{y}, \quad \text{or} \quad a : b :: x : y. \qquad \text{Q.E.D.}$$

CASE II. — When the magnitudes are *incommensurable.*



Let x and y be two incommensurable magnitudes, and a and b two equimultiples of x and y.

We are to prove $y : x :: b : a$.

Let x be divided into n equal parts, and designate the value of one of these parts by p. Let q denote the number of times x and y are contained in a and b respectively. Since x and y are incommensurable, p will be contained in y m times with a remainder r.

Then $\quad x = np$ (1) $\quad$ and $\quad qx = qnp$;

but $\quad a = qx, \quad \therefore\ a = qnp.$ (2)

Again, $\quad y = mp + r$ (3) $\quad$ and $\quad qy = qmp + qr$;

but $\quad b = qy, \quad \therefore\ b = qmp + qr.$ (4)

Dividing (4) by (2), and (3) by (1),

$$\frac{b}{a} = \frac{qmp}{qnp} + \frac{qr}{qnp}, \qquad \frac{y}{x} = \frac{mp}{np} + \frac{r}{np}.$$

Simplifying,

$$\frac{b}{a} = \frac{m}{n} + \frac{r}{np} \quad \text{and} \quad \frac{y}{x} = \frac{m}{n} + \frac{r}{np}.$$

Whence $\quad \dfrac{y}{x} = \dfrac{b}{a}$, or $\quad y : x :: b : a$. Q.E.D.

288. If two commensurable ratios can be expressed by the same numerical value, then these two ratios are equal, and consequently the four magnitudes composing those ratios are proportionals.

Let a, b, x, and y represent four magnitudes, a and b being commensurable, also x and y, so that, on applying their respective units of measure, $\dfrac{a}{b} = \dfrac{m}{n}$ and $\dfrac{x}{y} = \dfrac{m}{n}$.

Then $\dfrac{a}{b} = \dfrac{x}{y}$ (Axiom III.) or $a : b :: x : y$. Q.E.D.

289. If two incommensurable ratios can be expressed by the same approximate numerical value, however small the unit of measure, then these two ratios are equal, and the four magnitudes comprising those ratios are proportionals.

Post. Let $a:b$ and $x:y$ be two ratios, each of which is incommensurable and whose true values lie between the approximate values $\frac{m}{n}$ and $\frac{m}{n}+\frac{1}{n}$. (See 271.)

We are to prove that $a:b::x:y$; i.e. that there is no difference between the values of $\frac{a}{b}$ and $\frac{x}{y}$; i.e. that they are equal.

Dem. It is evident that $\frac{a}{b}$ and $\frac{x}{y}$ are either equal or unequal. Let us assume that they are unequal, and designate their difference by d. This difference being fixed and definite cannot be changed by any legitimate manipulation or transformation of the ratios.

By the hypothesis,

$$(1)\quad \frac{a}{b}>\frac{m}{n} \qquad \text{and} \qquad \frac{x}{y}>\frac{m}{n}.$$

$$(2)\quad \frac{a}{b}<\frac{m}{n}+\frac{1}{n} \qquad \text{and} \qquad \frac{x}{y}<\frac{m}{n}+\frac{1}{n}.$$

Consequently the difference between $\frac{a}{b}$ and $\frac{x}{y}$ must be less than $\frac{1}{n}$.

Now by decreasing the value of the common unit of measure, the value of n increases and that of $\frac{1}{n}$ decreases accordingly. Hence we may conceive n to be so great that $\frac{1}{n}$ will be smaller than d; i.e. the difference between $\frac{a}{b}$ and $\frac{x}{y}$ is *equal to* d and *less than* d at the same time. This is of course absurd or impossible. Hence they cannot differ in value; i.e. they are equal, and $a:b::x:y$. Q.E.D.

290. In the demonstration of the following theorems, the expression "four quantities" means the numerical measures of four geometrical magnitudes of the same kind. It would be well, however, to represent the magnitudes either by angles or lines, as in 287.

291. If four quantities form a proportion, the product of the means equals the product of the extremes.

292. *Remark.* Our ability to demonstrate the theorems in proportion depends on our knowledge of the principles governing the manipulation of the *equation.* Hence the first thing to be done is to change the *proportion form* to the equivalent *equation form.* The author is of the firm belief that mathematicians have no right to amalgamate these two *forms* of expression, and that pupils should be taught to rigidly discriminate in their use.

Post. Let the four quantities a, b, c, and d form a proportion so that

$$a : b :: c : d.$$

We are to prove that $ad = bc.$

Dem. $$\frac{a}{b} = \frac{c}{d}.$$

(Changing to *equation form.*)

$$\frac{ad}{b} = c.$$

(Multiplying both members of the equation by d.)

$$ad = bc.$$

(Multiplying both members of the previous equation by b.)

Q.E.D.

293. If three quantities form a proportion, the product of the extremes is equal to the square of the mean.

294. If the product of two quantities equals the product of two others, the two factors of either product may be made the means, and the other two the extremes of a proportion.

Post. Let $cx = an$.

We are required to form a proportion from the four factors c, x, a, and n, in which c and x shall be the extremes.

Dem. $$c = \frac{an}{x}. \qquad \text{Why ?}$$

$$\frac{c}{a} = \frac{n}{x}. \qquad \text{Why ?}$$

$$\therefore \quad c : a :: n : x.$$

(Changing from *equation form* to the *proportion form.*)

Form proportions from the following equations in which the factors of the product marked *Ex.* shall form the extremes, avoiding the factor 1.

I. $\overset{\text{Ex.}}{2x} = ac.$

II. $6 = \overset{\text{Ex.}}{ax}.$

III. $\overset{\text{Ex.}}{3\sqrt{a}} = 14.$

IV. $a(x + y) = \overset{\text{Ex.}}{n^2}.$

V. $y^2 = \overset{\text{Ex.}}{an + ac}.$

VI. $\overset{\text{Ex.}}{ab + bx} = cn + yc.$

VII. $ax + x = \overset{\text{Ex.}}{yb^2 + b^2}.$

VIII. $\overset{\text{Ex.}}{x^2 - 1} = a^2 - b^2.$

IX. $\overset{\text{Ex.}}{ax + xb + cx} = nd + hn + nk.$

X. $x^2 + 2cx + c^2 = \overset{\text{Ex.}}{an + ny}.$

295. If four quantities form a proportion, they will be in proportion by *inversion.*

Sug. Divide 1 by each member of the equation.

296. If four quantities form a proportion, they will be in proportion by *alternation.*

Post. Let the four quantities a, x, c, and n form a proportion so that

$$a : x :: c : n.$$

We are to prove that

$$a : c :: x : n.$$

Dem. $$\frac{a}{x} = \frac{c}{n}.$$ Why?

$$\frac{a}{xc} = \frac{1}{n}.$$ Why?

$$\frac{a}{c} = \frac{x}{n}.$$ Why?

$$\therefore \quad a : c :: x : n.$$ Q.E.D.

Query. If the corresponding terms of two proportions be *added,* will the sums form a proportion? *Numerical example.*

297. If four quantities form a proportion, they will be in proportion by *composition.*

Post. Let the four quantities x, y, a, and b form a proportion so that

$$x : y :: a : b.$$

We are to prove,

$$\text{I.} \quad x + y : y :: a + b : b.$$

$$\text{II.} \quad x + y : x :: a + b : a.$$

Dem. $$\frac{x}{y} = \frac{a}{b}.$$ Why?

$$\frac{x}{y}+1=\frac{a}{b}+1. \qquad \text{Why?}$$

$$\frac{x}{y}+\frac{y}{y}=\frac{a}{b}+\frac{b}{b}. \qquad \text{Why?}$$

$$\frac{x+y}{y}=\frac{a+b}{b}. \qquad \text{Why?}$$

$$\therefore \quad x+y:y::a+b:b. \qquad \text{Why?}$$

Q.E.D.

The pupil should demonstrate Part II.

Sug. Consult 295.

298. If four quantities form a proportion, they will be in proportion by *division.*

Sug. There are four cases. In Case I. subtract 1 from each member of the equation. The other cases may be similarly demonstrated by reversing the above process, and consulting 295.

299. If two proportions have a ratio in each equal, the other two ratios will form a proportion.

300. If two proportions have the two antecedents of one equal respectively to the two antecedents of the other, the consequents will form a proportion.

301. If two proportions have the two consequents of one equal respectively to the two consequents of the other, the antecedents will form a proportion.

302. If four quantities form a proportion, they will be in proportion by *composition and division.*

Sug. Consult 298, 297, 296, and 299.

303. If any number of magnitudes of the same kind form a proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Post. Let the quantities a, x, c, n, d, and r form a continued proportion, so that

$$a : x :: c : n :: d : r.$$

We are to prove,

$$\text{I.} \quad a + c + d : x + n + r :: a : x;$$

or $$\text{II.} \quad :: c : n;$$

or $$\text{III.} \quad :: d : r.$$

Dem. Now, if this theorem can be demonstrated, then Case I. must be a true proportion. Let us temporarily assume it to be true, and trace the results.

First. $\quad x(a + c + d) = a(x + n + r),$ Why?

or $\quad ax + cx + dx = ax + an + ar;$ Why?

but $\quad (1) \quad ax = ax.$ Why?

$\therefore \quad cx + dx = an + ar.$ Why?

$(2) \quad cx = an.$ Why?

$\therefore \; (3) \quad dx = ar.$ Why?

Now it is evident that if we could obtain the equations (1), (2), (3) from our given proportions, we could reverse the above process, and thus demonstrate the theorem.

Let us bear in mind that the ratio $a : x$ in Case I. is the one that is to form the proportion with that composed of the sums of the antecedents and consequents, and that equation (1) is formed from the product of the terms of that ratio.

Hence $\quad (1) \quad ax = ax;$ Why?

$(2) \quad cx = an;$ Why?

$(3) \quad dx = ar.$ Why?

Hence $\quad ax + cx + dx = ax + an + ar,$ Why?

or $\quad x(a + c + d) = a(x + n + r).$ Why?

The pupil should be able to finish this case, and also demonstrate the other two without difficulty. Consult 204.

304. If four quantities form a proportion, the terms of either ratio may be either multiplied or divided by the same quantity, and the results still form a proportion.

304 (*a*). If the antecedents or consequents of a proportion be either multiplied or divided by the same quantity, the results will still form a proportion.

305. If four quantities form a proportion, and the terms of one ratio be either multiplied or divided by the same quantity, while both terms of the other ratio be either multiplied or divided, either by the same or different quantity from that used in the first ratio, the results will still form a proportion.

Sug. The pupil should take each case involved in the statement of the above theorem separately.

306. If two proportions be given, the products of the corresponding terms will also form a proportion.

307. If two proportions be given in which two corresponding ratios have the antecedent of one equal to the consequent of the other, the remaining antecedent and consequent, together with the products of the corresponding terms of the other two ratios, will form a proportion.

308. If the antecedents of a proportion are equal, the consequents are equal.

308 (*a*). Converse of 308.

309. If four quantities form a proportion, their like powers and like roots will also form a proportion.

310. If three quantities form a proportion, the first is to the third as the square of the first is to the square of the second.

311. If four quantities form a proportion, the sum of the squares of the first two terms is to their product as the sum of the squares of the last two is to their product.

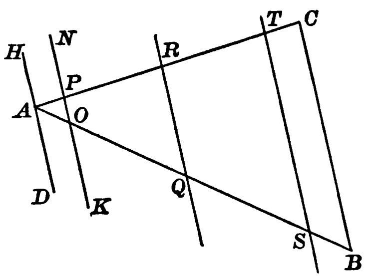
311 (*a*). Substitute "difference" for "sum" in 311.

312. If two quantities be either increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.

312 (*a*). If three terms of a simple proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms of the two proportions are equal.

PROPORTIONAL LINES.

313.* If a line be drawn parallel to one side of a triangle, the four parts into which it divides the other sides will form a proportion.



Post. Let ABC be any triangle, and ST a line drawn parallel to the side CB.

We are to prove that the four parts AS, SB, AT, and TC form a proportion.

Dem. Let us conceive the line DH passing through the vertex A and parallel to BC, to move toward BC and remaining always parallel to it. The instant it starts there will, of course, be two points of intersection, as O and P. It is evident that when the point O has reached the point B, the point P will have reached the point C. Why?

Again, if Q and R be the middle points of the sides AB and AC respectively, it is evident that when the point O reaches the point Q, the point P will reach the point R. Why?

Hence when the point O has moved over one-half of AB, the point P has moved over one-half of AC. Similarly, when O has moved over one-fourth, one-tenth, one-thousandth, or

* See Appendix.

one-*n*th of AB, P has moved over the same part of AC. Hence, no matter what the position of the moving line, AO is always the same part of AB that AP is of AC; that is, the ratio of AO to AB is the same as the ratio of AP to AC; or

$$AO : AB :: AP : AC.$$

Now, since by Hyp. ST is parallel to BC when point O reaches S, point P reaches T.

Hence $\quad AS : AB :: AT : AC.$

But $\quad AB - AS : AS :: AC - AT : AT.$ Why?

But $\quad AB - AS = SB$ and $AC - AT = TC.$

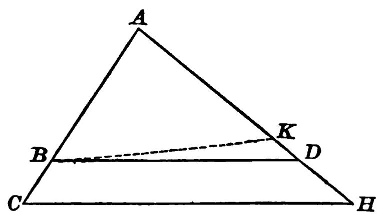
$\therefore\ SB : AS :: TC : AT.$ Why?

Q.E.D.

314. If a line be drawn parallel to one side of a triangle, the other two sides and either pair of corresponding parts form a proportion.

Sug. Use result obtained in previous theorem, and consult 297.

315. Converse of 313.



Post. Let ACH be a triangle, with line BD drawn so that

$$AB : BC :: AD : DH.$$

We are to prove BD parallel to CH.

Dem. First, suppose BK to be drawn through B parallel to CH.

Then $\quad AB : BC :: AK : KH.$ Why?

$\therefore\ AD : DH :: AK : KH.$ Why?

$\therefore\ AD + DH : DH :: AK + KH : KH;$ Why?

or $\quad AH : DH :: AH : KH.$

$\therefore\ DH = KH.$ Why?

$\therefore$ points D and K must coincide. Why?

What, then, must be the relative position of the lines *BK* and *BD*? Why?

But *BK* was drawn parallel to *CH*. Consequently *BD*, which coincides with *BK*, must also be parallel to *CH*. Q.E.D.

316. If a series of parallel transversals, intersecting any two straight lines, intercept equal distances on one of these lines, they will also intercept equal distances on the other.

317. If a series of parallel transversals, intersecting any number of straight lines, intercept equal distances on one of these lines, they will also intercept equal distances on each of the others.

318. If two lines be drawn parallel to one side of a triangle intersecting the other two sides, the parts thus intercepted form a proportion.

318 (*a*). If a line be drawn parallel to one side of a triangle, this side, its parallel, and either of the other two sides, together with the segment of the latter joining the third side, will form a proportion.

319. If any number of straight lines be intersected by a series of parallel transversals, the parts thus intercepted by the latter form a proportion.

320. If a line be drawn parallel to the base of a triangle, and another from vertex to base, the parts of both the base and its parallel will form a proportion.

321. If any number of straight lines which pass through a common point be intersected by a series of parallel transversals, the parts intercepted by both parallels and non-parallels form a proportion.

322. Converse of 321.

323. The bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.

Sug. From the vertex of the bisected angle extend one of

the sides, and from one of the other vertices draw a line parallel to the bisector meeting the side extended.

324. If the bisector of an exterior angle of a triangle meet one of the sides extended, this side plus its extension, the extension, and the other two sides of the triangle, form a proportion.

Sug. From extremity of the side extended, draw a parallel to the bisector.

POLYGONS.

325. A *polygon* is a portion of a plane bounded by straight lines.

What is the least number of sides that a polygon can have? The greatest number?

326. The word "*polygon*" is from two Greek words meaning *many angles.*

What is a polygon of three sides usually called? It is also called a *trigon.* A polygon of four sides is usually termed what? It is also called a *tetragon.* A polygon of five sides is called a *pentagon,* one of six sides a *hexagon,* one of seven sides a *heptagon,* one of eight sides an *octagon,* one of nine sides a *nonagon* or *enneagon,* one of ten sides a *decagon,* one of eleven sides an *undecagon,* one of twelve sides a *dodecagon,* and one of fifteen sides a *pentedecagon.*

327. A *convex polygon* is one each of whose angles is less than 180°.

A *concave polygon* is one that has one or more re-entrant angles; i.e. where, at one or more vertices, the angular space within the polygon is more than 180°.

Whenever polygons are mentioned in this work, convex polygons are meant unless otherwise specified.

The *diagonal* of a polygon is a line joining any two vertices not contiguous.

328. A *regular polygon* is one that is both equilateral and equiangular.

The *perimeter* of a polygon is the sum of its sides. How many angles has each of the above-named polygons? How does the number of sides compare, in each case, with the number of angles? How many angles, then, has a polygon of n sides? How many diagonals can be drawn from a single vertex in each of the above-named polygons? Compare the number of diagonals in each case with the number of sides. If the polygon has n sides, how many diagonals can be drawn from a single vertex? How many triangles are formed by drawing diagonals from a single vertex in each of the above-named polygons? How does the number of triangles compare with the number of sides? If the polygon has n sides, how many triangles would be formed by diagonals similarly drawn?

If from any point selected at random in a polygon lines be drawn to the vertices, into how many triangles will it be thus divided?

329. The sum of the interior angles of a polygon is equal to two right angles, taken as many times, less two, as the polygon has sides; or twice as many right angles as the polygon has sides, less four right angles.

Sug. Divide the polygon up into triangles, and let n equal the number of sides. Having found an expression in terms of n, and *a right angle* for the sum of the angles, what would be the value of *one* angle if the polygon were *equiangular?*

330. The sum of the exterior angles of any polygon formed by extending each of its sides in succession similarly, is equal to four right angles.

Sug. Consult Sect. 25 and Theorem 325.

Problems of Computation.

331. Find the value in right angles of the sum of the interior angles of a *pentagon, hexagon, octagon, decagon, dodecagon, pentedecagon,* and a polygon of *fifty-two* sides.

332. Find the value, in terms of a right angle as the unit, of one of the angles of an *equiangular pentagon, octagon, dodecagon,* a polygon of *twenty* sides, *one hundred* sides.

333. Find the values of the above angles in degrees, also in grades.

334. How many sides has an equiangular polygon, the sum of four angles of which is equal to seven right angles?

335. How many sides has an equiangular polygon, the sum of three angles of which is equal to five right angles?

336. How many sides has an equiangular polygon, the sum of nine angles of which is equal to sixteen right angles?

337. How many sides has the polygon, the sum of whose interior angles is equal to the sum of its exterior angles?

338. How many sides has the polygon, the sum of whose interior angles is double that of its exterior angles?

339. How many sides has the polygon, the sum of whose exterior angles is double that of its interior angles?

340. How many sides has the polygon, the sum of whose interior angles is equal to nine times the sum of its exterior angles?

341. How many sides has the equiangular polygon, when the sum of nine of its interior angles is four times the sum of its exterior angles?

342. How many sides has the equiangular polygon, when the sum of five of its interior angles is equal to two and one-fourth times the sum of its exterior angles?

343. How many sides has the equiangular polygon when

(*a*) $5 \measuredangle s = 9\frac{3}{5}$ rt. $\measuredangle s$?	(*e*) $8 \measuredangle s = 3\frac{7}{8}$ times its ext. $\measuredangle s$?
(*b*) 4 " $= 7\frac{1}{2}$ "	(*f*) 6 " $= 3\frac{3}{4}$ " " "
(*c*) 16 " $= 31$ "	(*g*) 7 " $= 6\frac{2}{3}$ rt. $\measuredangle s$?
(*d*) 5 " $= 8$ "	(*h*) 3 " $= 5\frac{2}{3}$ "

SIMILAR FIGURES.

344. Similarity in general means *likeness of form;* i.e. two figures are said to be similar when they have the *same shape,* although they may differ in size.

Similar geometrical figures are those whose homologous angles are equal, and whose homologous sides form a proportion.

345. *Homologous sides* of similar polygons are those which join the vertices of equal angles in the respective polygons.

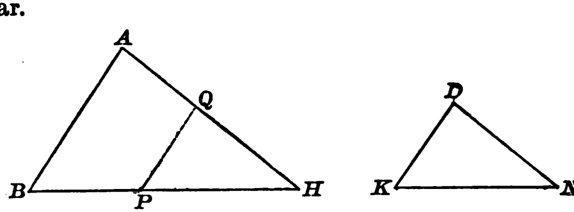
In similar triangles the homologous sides are those that are *opposite* the equal angles.

The pupil should be careful to observe at the outset that similarity involves two things; viz. *equality of angles* and *proportionality of sides.*

346. Two polygons are said to be *mutually equiangular* when the angles of one are equal respectively to the corresponding angles of the other, and *mutually equilateral* when each side of one has an equal side in the other.

SIMILAR TRIANGLES.

347. If two triangles be mutually equiangular, they are similar.



Post. Let ABH and KDN be two triangles having

$$\angle A = \angle D, \quad \angle B = \angle K, \quad \text{and} \quad \angle H = \angle N.$$

We are to prove the triangles similar; i.e. that their homologous sides form a proportion.

Cons. Mark off on HB a distance HP equal to KN, and on HA a distance HQ equal to DN, and join PQ.

Dem. What relation exists between the two $\triangle$ RDN and PQH? Why?

What, then, must be the relation between the $\angle$ PQH and D? What between $\angle$ QPH and K?

What, then, must be the relation between the $\angle$ PQH and A? What between $\angle$ QPH and B? Why?

What, then, must be the relative position of the lines AB and QP? Why?

Apply Theorem 314, and the remainder of the demonstration should be easy. Q.E.D.

348. If two triangles have two angles of one equal respectively to two angles of the other, the two triangles are similar.

349. If two right triangles have an acute angle of one equal to an acute angle of the other, the two triangles are similar.

350. If two triangles have an angle in each equal, and the sides including those angles form a proportion, the two triangles are similar.

Sug. Apply one to the other so that the equal angles shall coincide; then consult Theorem 315. Or proceed in a manner similar to that in Theorem 347.

351. If two triangles have their sides respectively parallel, they are similar.

Sug. Consult 94 and 98, then three possible hypotheses regarding the relations of the respective pairs of angles.

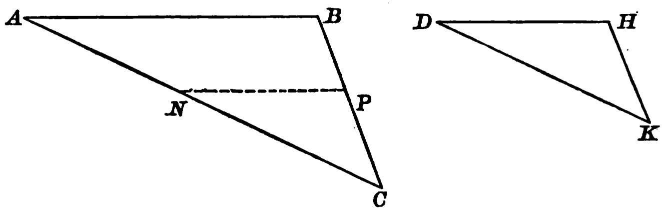
352. If two triangles have their sides respectively perpendicular to each other, they are similar.

Sug. Consult 95 and 98, and see suggestions to 351. Consult the *caution* in Sect. 55.

353. If two triangles be similar, their homologous altitudes are in the same ratio as either pair of homologous sides including the vertical angle.

354. If two triangles be similar, their homologous altitudes are in the same ratio as their homologous bases.

355. If the homologous sides of two triangles form a proportion, the triangles are similar.



Post. Let ABC and DHK be two triangles, their homologous sides forming the continued proportion.

$$AB : DH :: AC : DK :: BC : HK.$$

We are to prove that the two triangles are similar; i.e. that they are mutually equiangular.

Cons. Make NC equal to DK, PC equal to HK, and join NP.

Dem. In the given proportion substitute for DK and HK their equals NC and PC. Then

$$AC : NC :: BC : PC.$$

What is true, then, of the two triangles ABC and NPC? See Theorem 350.

$$\therefore\ AB : NP :: BC : PC. \qquad \text{Why?}$$

But by Hyp.,

$$AB : DH :: BC : HK.$$

What is true of the first terms of these two proportions? Of the third terms? Of the last terms? What must be true, then, of the second terms?

The pupil should finish the demonstration without difficulty.

356. If two polygons be similar, the diagonals drawn from homologous vertices will divide them into the same number of triangles, similar two and two, and similarly placed.

357. Converse of 356.

358. The perimeters of two similar polygons are in the same ratio as any two homologous sides or any two homologous diagonals.

359. The perimeters of two similar polygons are in the same ratio as the bisectors of any two homologous angles or any two homologous lines howsoever drawn.

360. If, in a right triangle, a perpendicular be drawn to the hypothenuse from the vertex of the right angle,

I. the two triangles thus formed are similar to the original triangle and to each other.

II. the perpendicular and the two segments of the hypothenuse form a proportion.

(Where three quantities form a proportion, what relation must one of these quantities sustain to the other two?)

III. either leg, the hypothenuse, and that segment of the latter which joins the former, form a proportion.

(The pupil should preserve these proportions for use further on.)

IV. the squares of the two legs and the two segments of the hypothenuse form a proportion.

Sug. Use the two proportions in III. by Theorem 293, and divide.

V. the square of the hypothenuse bears the same ratio to the square on either leg, as the hypothenuse bears to that segment of the latter joining that leg.

VI. the two legs, the hypothenuse, and the perpendicular form a proportion.

360 (*a*). The square of the hypothenuse of a right triangle is equal to the sum of the squares of the two legs. See Theorem 426.

Sug. Use result in 360, IV., by composition, and then compare with 360, V.

361. The square of the diagonal of a square is equal to twice the square of one of its sides.

PROJECTION.

362. The projection of one line upon another is that portion of the latter included between two perpendiculars to the latter drawn from the extremities of the former.

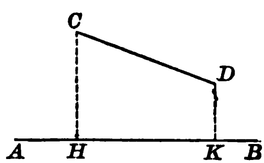


Fig. I.

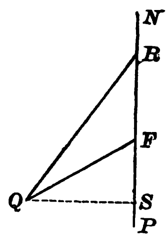


Fig. II.

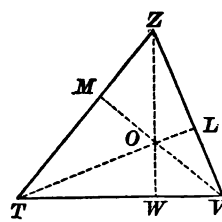


Fig. III.

For example, HK is the projection of the line CD upon AB, and RS is the projection of RQ on NP; it is also the projection of RQ on RF. Mention all the cases of projection in Figure III., the dotted lines being perpendicular to the respective sides.

363. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of those sides and the projection of the other upon that side.

If C be the acute angle, then by a glance at the following diagram it will be readily seen that there may be two cases depending upon whether the projection involves an extension

of one side, which will evidently be the case if the side to be projected is opposite an obtuse angle.

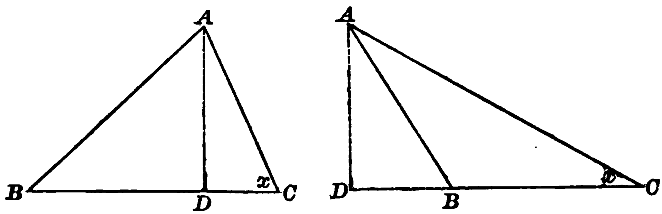


Fig. I. Fig. II.

We are to prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \cdot DC.$$

In Case I. $DB = BC - DC.$

In Case II. $DB = DC - BC.$

The square of either of these two equations gives the same result; hence to the square add the squares of the perpendicular, and then combine the terms by using Theorem 360 (a).

364. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of those sides and the projection of the other upon that side.

Sug. Form an equation by placing the projection of the side opposite the obtuse angle equal to the sum of its two parts, then proceed as in 363.

365. If any median of a triangle be drawn,

I. the sum of the squares of the other two sides is equal to twice the square of one-half the bisected side, plus twice the square of the median; and

II. the difference of the squares of the other two sides is equal to twice the product of the bisected side and the projection of the median upon that side.

Sug. Use 363 and 364, then combine the resulting equations.

366. In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, plus four times the square of the line joining the middle points of the diagonals.

Sug. Use 365. What modification of the above theorem would result if the quadrilateral were a parallelogram?

367. If from any point, selected at random, in the circumference of a circle, a perpendicular be drawn to a diameter, this perpendicular will be a mean proportional between the two segments of the diameter.

368. If two chords of a circle intersect each other, the four parts form a proportion.

369. If two secants be drawn from the same point without the circle, the entire secants and the parts that are without the circle form a proportion.

370. If, from the same point, a tangent and secant to a circle be drawn, the tangent, the secant, and that part of the latter that is outside the circle, will form a proportion.

371. The two tangents to two intersecting circles from any point in their common secant are equal.

Sug. Consult 370.

372. If two circles intersect each other, their common chord extended bisects their common tangents.

Advance Theorems.

373. In any triangle the product of any two sides is equal to the diameter of the circumscribed circle multiplied by the perpendicular drawn to the third side from the vertex of the angle opposite.

Sug. Construct the diameter from the same vertex as the perpendicular, and join its extremity with one of the other vertices, making a right triangle similar to the one formed by the perpendicular, whence the necessary proportion.

374. In any triangle the product of any two sides is equal to the product of the segments of the third side, formed by the bisector of the opposite angle, plus the square of the bisector.

Sug. Circumscribe a circle, and extend the bisector to the circumference, and connect its extremity with one of the other vertices of the triangle. Then if the vertices of the triangle be lettered A, B, and C, and the bisector be drawn from A, and D the point where the bisector crosses the side BC, and E the extremity of the bisector extended, then the triangles BAD and ACE can be proved similar.

375. If one side of a right triangle is double the other, the perpendicular, from the vertex of the right angle to the hypothenuse, divides it into segments which are to each other as 1 to 4.

376. A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals.

377. In any triangle the product of any two sides is equal to the product of the segments of the third side, formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.

Sug. Consult Theorem 374.

378. The perpendicular, from the intersection of the medians of a triangle, upon any straight line in the plane of the triangle, is one-third the sum of the perpendiculars from the vertices of the triangle upon the same line.

379. If two circles are tangent to each other, their common tangent and their diameters form a proportion.

380. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller.

381. In any quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.

Sug. From one vertex draw a line to the opposite diagonal, making the angle formed by it and one side equal to the angle formed by the other diagonal and side which meets the former.

382. If three circles whose centres are not in the same straight line intersect one another, the common chords will intersect each other at one point.

383. If two chords be perpendicular to each other, the sum of the squares of the four segments is equal to the square of the diameter.

384. The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines joining the middle points of the opposite sides.

Problems of Computation.

385. The chord of one-half a certain arc is 9 inches, and the distance from the middle point of this arc to the middle of its subtending chord is 3 inches. Find the diameter of the circle.

386. The external segments of two secants to a circle from the same point are 10 inches and 6 inches, while the internal segment of the former is 5 inches. What is the internal segment of the latter?

387. The hypothenuse of a right triangle is 16 feet, and the perpendicular to it from the vertex of the opposite angle is 5 feet. Find the values of the legs and the segments of the hypothenuse.

388. The sides of a certain triangle are 6, 7, and 8 feet respectively. In a similar triangle the side corresponding to 8 is 40. Find the other two sides.

389. The sides of a certain triangle are 9, 12, and 15 feet respectively. Find the segments of the sides made by the bisectors of the several angles.

390. If a vertical rod 6 feet high cast a shadow 4 feet long, how high is the tree which, at the same time and place, casts a shadow 90 feet long?

391. The perimeters of two similar polygons are 200 and 300 feet respectively, and one side of the former is 24 feet. What is the corresponding side of the latter?

392. How long must a ladder be to reach a window 24 feet high, if the lower end of the ladder is 10 feet from the side of the house?

393. Find the lengths of the longest and the shortest chord that can be drawn through a point 6 inches from the centre of a circle whose radius is 10 inches.

394. The distance from the centre of a circle to a chord 10 inches long is 12 inches. Find the distance from the centre to a chord 24 inches long.

395. The radius of a circle is 5 inches. Through a point 3 inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance from one end of the chord to the ends of the diameter.

396. Through a point 10 feet from the centre of a circle whose radius is 6 feet tangents are drawn. Find the lengths of the tangents and of the chord joining the points of contact.

397. If a chord 8 feet long be 3 feet from the centre of the circle, find the radius and the distances from the end of the chord to the ends of the diameter which bisects the chord.

398. Through a point 5 inches from the centre of a circle whose radius is 13 inches any chord is drawn. What is the

product of the two segments of the chord? What is the length of the shortest chord that can be drawn through that point?

399. From the end of a tangent 20 inches long a secant is drawn through the centre of a circle. If the exterior segment of this secant be 8 inches, what is the radius of the circle?

400. A tangent 12 feet long is drawn to a circle whose radius is 9 feet. Find the external segment of a secant through the centre from the extremity of the tangent.

401. The span of a roof is 28 feet, and each of its slopes measures 17 feet from the ridge to the eaves. Find the height of the ridge above the eaves.

402. A ladder 40 feet long is placed so as to reach a window 24 feet high on one side of the street, and on turning the ladder over to the other side of the street, it just reaches a window 32 feet high. What is the width of the street?

403. The bottom of a ladder is placed at a point 14 feet from a house, while its top rests against the house 48 feet from the ground. On turning the ladder over to the other side of the street its top rests 40 feet from the ground. Find the width of the street.

404. One leg of a right triangle is 3925 feet, and the difference between the hypothenuse and other leg is 625 feet. Find the hypothenuse and the other leg.

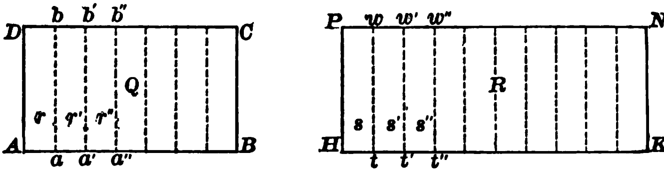
AREAS.

405. The *area* of a surface is its numerical measure; i.e. the numerical expression for the number of times it contains another surface arbitrarily assumed as a unit of measure. For example, the area of the floor of a room is the number of times it contains some one of the common units of surface, square foot, square yard, etc. This unit of surface is called the *superficial unit.* The most convenient superficial unit is

the square of the linear unit. The square of a linear unit (or line) is the surface of the square constructed with that linear unit for its sides. Surfaces that are equal in area are said to be *equivalent.*

406.* If two rectangles have equal altitudes, their areas will be in the same ratio as their bases.

CASE I. — When the bases are *commensurable* (262).



Post. Let $ABCD$ and $HKNP$ be two rectangles, having their altitudes AD and HP equal, and their bases AB and HK commensurable. Designate their areas by Q and R respectively.

We are to prove $\quad Q : R :: AB : HK.$

Dem. Since AB and HK are commensurable, they must have a common measure (262).

Suppose this common measure to be contained in AB n times, and in HK m times.

At the several points of division, a, a', a'', etc., t, t', t'', etc., construct ab, $a'b'$, $a''b''$, etc., perpendicular to AB, and tw, $t'w'$, $t''w''$, etc., perpendicular to HK.

Then the rectangle $ABCD$ will be divided into n equal rectangles r, r', r'', etc., and the rectangle $HKNP$ will be divided into m equal rectangles s, s', s'', etc. These smaller rectangles are also equal to one another.

(The pupil should demonstrate the above propositions.)

Hence $\quad AB : HK :: n : m, \quad (264.)$

and $\quad Q : R :: n : m. \quad (264.)$

$\therefore \; Q : R :: AB : HK. \quad (288.)$ Q.E.D.

* See Appendix.

CASE II. — When the bases are *incommensurable.*

Suppose AB to be divided into any number of equal parts, as n, and that one of these parts be applied to HK and found to be contained m times with a remainder less than that part. From the points of division construct lines as in Case I. Then the rectangle $ABCD$ will be divided into n equal rectangles, and rectangle $KNPH$ into m equal rectangles, with a remaining rectangle less than one of the m rectangles.

Then $Q:R$ and $AB:HK$ are two incommensurable ratios.

$$(1)\quad \frac{R}{Q} > \frac{m}{n} \quad \text{and} \quad \frac{R}{Q} < \frac{m}{n} + \frac{1}{n}.$$

$$(2)\quad \frac{HK}{AB} > \frac{m}{n} \quad \text{and} \quad \frac{HK}{AB} < \frac{m}{n} + \frac{1}{n}.$$

Hence the two ratios $R:Q$ and $HK:AB$ have the same approximate numerical value, whatever the magnitude of n.

$$\therefore \quad R:Q::HK:AB, \quad (289.)$$

or $$Q:R::AB:HK.$$ Q.E.D.

407. If two rectangles have equal bases, their areas are in the same ratio as their altitudes.

Sug. Consider their altitudes as bases, and bases as altitudes, and proceed as in 406.

408. The areas of any two rectangles have the same ratio as the products of their respective bases and altitudes.

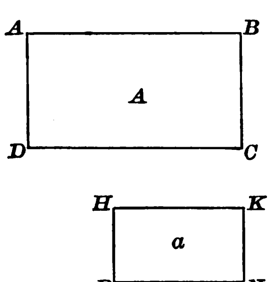


Fig. I.

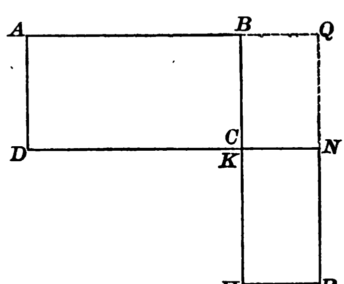


Fig. II.

(By the product of two lines is meant the product of the numbers which represent them when both are measured by the same linear unit.)

Post. Let $ABCD$ and $HKNP$ be any two rectangles, with DC and PN their respective bases. Let us designate their areas by A and a respectively, their altitudes by H and h, and their bases by B and b.

We are to prove that $A : a :: B \times H : b \times h$.

Dem. Consider the two rectangles so placed that the vertices C and K shall coincide, and the four angles all equal.

Then BCH is one straight line, as is also DKN. Why?

Complete the rectangle $BQNC$. Then considering CN the base of rectangle $BQNC$, it has the same altitude as rectangle $ABCD$.

$\therefore$ Area $ABCD$: Area $BQNC :: DC : KN$. (1) Why?

Again, considering BC the base of the rectangle $BQNC$,

Area $HKNP$: Area $BQNC :: HK : BC$, (2)

or Area $BQNC$: Area $HKNP :: BC : HK$. (3) Why?

Hence

Area $ABCD$: Area $HKNP :: BC \times DC : HK \times KN$,

(See Theorem 307.)

or $A : a :: B \times H : b \times h$. Q.E.D.

408 (a). Hence the above demonstration shows that if the bases and altitudes of any two rectangles be measured by the same linear unit, the ratio of their respective products (see 408) will be the ratio of their respective surfaces, and consequently either may be assumed as the measure of the other.

For example, suppose the linear unit to be contained 5 times in AD, 18 times in DC, 3 times in HP, and 6 times in PN.

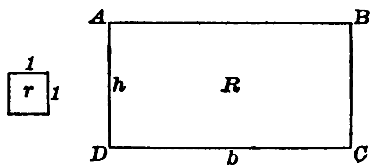
Then $A : a :: 5 \times 18 : 3 \times 6$,

or $$\frac{A}{a} = \frac{90}{18}.$$

This simply means that, using a as the unit, the area A is 5 times as large as the area a; or that the area a, using A as the unit surface, is $\frac{1}{5}$ as large as A.

Practically it is more convenient to compare the areas of both rectangles with the square of the linear unit (405) as a unit of surface. This comparison is formally shown in the enunciation and demonstration of the following theorem.

409. The area of a rectangle is measured by the product of its base and altitude.



Post. Let $ABCD$ be any rectangle; and in whatever linear unit the base and altitude be expressed, let r be a square whose sides are the same unit. Designate its area by R, and its base and altitude by b and h respectively.

We are to prove $\quad R = b \times h.$

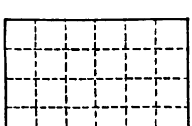
Dem. $\quad R : r :: b \times h \quad 1 \times 1, \quad (408.)$

or
$$\frac{R}{r} = \frac{b \times h}{1 \times 1};$$

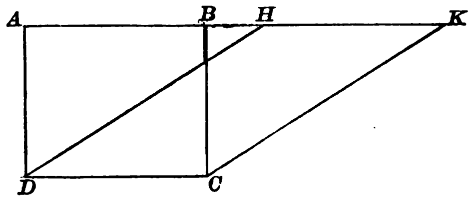
or, since r is the *unit* of area,

$$R = b \times h. \qquad \text{Q.E.D.}$$

When the base and altitude are commensurable, this is rendered evident by dividing the rectangle into squares, each equal to the superficial unit, as shown in annexed figure. The above demonstration, however, includes the case when the base and altitude are incommensurable.



410. If a parallelogram and rectangle have the same or equal bases and the same or equal altitudes, they are equivalent.



Sug. Place them so that their bases shall coincide, as represented in above diagram. Then AB and HK are in the same straight line. Why?

Prove equality of the triangles ADH and BCK, and apply Axiom VII.

411. The area of a parallelogram is measured by the product of its base and altitude.

412. If two parallelograms have the same or equal bases and the same or equal altitudes, they are equivalent.

413. If two parallelograms have the same or equal bases, their areas are in the same ratio as their altitudes.

414. If two parallelograms have the same or equal altitudes, their areas are in the same ratio as their bases.

415. The areas of any two parallelograms are in the same ratio as the products of their respective bases and altitudes.

416. If a triangle and parallelogram have the same or equal bases and the same or equal altitudes, the area of the latter is double that of the former.

Sug. Place them so that their bases will coincide.

417. The area of a triangle is measured by one-half the product of its base and altitude.

418. If two triangles have the same or equal bases and the same or equal altitudes, they are equivalent.

419. If two triangles have the same or equal bases, their areas are in the same ratio as their altitudes.

420. If two triangles have the same or equal altitudes, their areas are in the same ratio as their bases.

421. The areas of any two triangles are in the same ratio as the products of their respective bases and altitudes.

422. The area of a trapezoid is equal to the product of its altitude and one-half the sum of its parallel sides.

423. The square described upon the sum of two lines is equivalent to the sum of the squares upon the two lines, plus twice the rectangle formed by the two lines.

424. The square described on the difference of two lines is equivalent to the sum of the squares upon the two lines, minus twice the rectangle formed by the two lines.

425. The rectangle formed by the sum and difference of two lines is equivalent to the difference of the squares upon the two lines.

426. The square described upon the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the legs.

This theorem was first demonstrated by Pythagoras, about 450 B.C., and hence is called the *Pythagorean* theorem. It has been a favorite one with mathematicians, and consequently *about fifty* different demonstrations of it have been recorded. Among the many diagrams used, the following are a few; and it is hoped that the pupil will endeavor to demonstrate it for himself, using either one of these, or, preferably, one of his own. No. I. is the diagram used by Euclid in the demonstration of this theorem, constituting his famous "47th." No. VII. is the diagram used by the late President Garfield, who was said to have utilized his leisure hours in Congress in mathematical investigation. In each figure *ABC* is the given triangle, and in No. VII. *ADC* is one-half the square upon the hypothenuse. *DH* is drawn parallel to *BC* to meet *BA* extended. The method is algebraic, and involves an equation between the sum of the areas of the three triangles and the trapezoid *BCDH*.

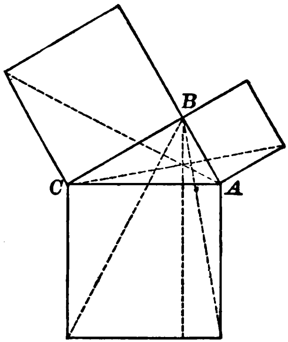


Fig. I.

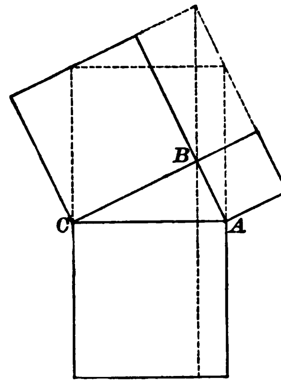


Fig. II.

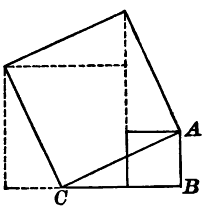


Fig. III.

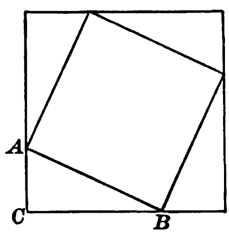


Fig. IV.

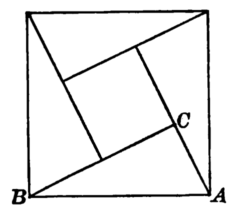


Fig. V.

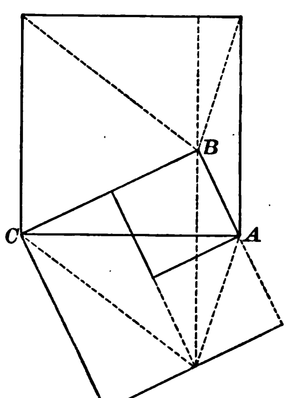


Fig. VI.

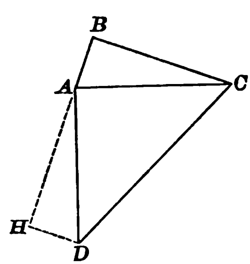


Fig. VII.

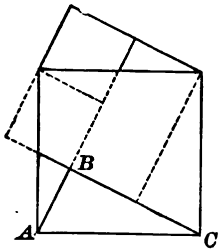


Fig. VIII.

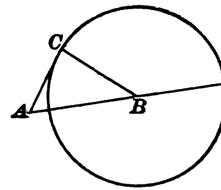
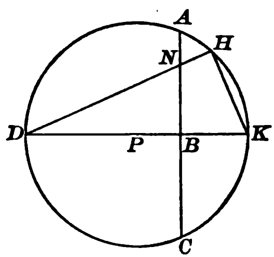


Fig. IX.

See also Theorem 360 (*a*).

427. The following demonstration is original with the author:



Post. Let AC be any chord in the circle DAC, and DK a diameter perpendicular to AC. Also let DH be drawn intersecting AC. Join HK; then

$$DN : DK :: DB : DH,$$

$$\therefore DN \times DH = DK \times DB;$$

but $\quad DK = DB + BK;$

hence $\quad DN \times DH = DB\,(DB + BK).$

$$DN \times DH = \overline{DB}^2 + DB \times BK;$$

but $\quad DB \times BK = AB \times BC = \overline{AB}^2,$

$$\therefore DN \times DH = \overline{DB}^2 + \overline{AB}^2.$$

Now conceive chord DH to revolve about D as a centre until the point H coincides with point A;

then $$DN = DH = DA.$$

$$\therefore \overline{DA}^2 = \overline{DB}^2 + \overline{AB}^2. \qquad \text{Q.E.D.}$$

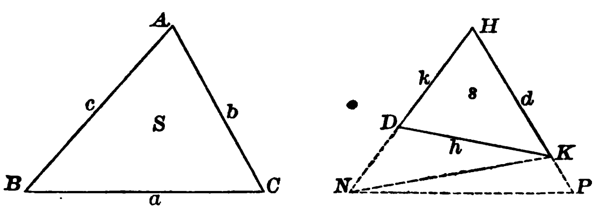
For other methods of demonstrating this theorem, see *Journal of Education*, June 23 and July 7, 1887; and May 24 and June 28, 1888.

427 (a). Converse of 426.

428. The diagonal of a square is incommensurable with its side.

Sug. Find value of diagonal in terms of its side by Theorem 426.

429. If two triangles have an angle in each equal, their areas are in the same ratio as the products of the sides which include the equal angles.



Post. Let ABC and DHK be two triangles with

$$\angle A = \angle H.$$

Designate their areas by S and s, and the sides opposite the several angles by the corresponding small letters.

We are to prove that

$$S : s :: c \times b : k \times d.$$

Apply the triangles so that the equal angles shall coincide, or extend the sides of one until they equal the corresponding sides of the other, as in the above diagram.

Dem. I. Area DHK : Area NHK :: DH : NH. Why?

II. Area NHK : Area NHP :: HK : HP. Why?

Whence

Area DHK : Area NHP :: $DH \times HK$: $NH \times HP$.

Why?

or $S : s :: k \times d : c \times b$. Q.E.D.

430. The areas of two similar triangles have the same ratio as the squares of any two homologous sides.

Sug. Consult 421 and 354.

431. The areas of two similar polygons are in the same ratio as the squares of any two homologous sides.

Sug. Consult 356, 430, and 303.

432. The areas of two similar polygons are in the same ratio as the squares of any two homologous diagonals.

433. Any two homologous sides or diagonals of two similar polygons are in the same ratio as the square roots of their areas.

434. If similar polygons are described upon the sides of a right triangle as homologous sides, the polygon described upon the hypothenuse is equivalent to the sum of the polygons upon the other two sides.

Advance Theorems.

435. If two triangles have two sides of one equal respectively to two sides of the other, and their included angles supplementary, the triangles are equivalent.

436. If a straight line be drawn through the centre of a parallelogram, the two parts are equivalent.

437. If through the middle point of the median of a trapezoid a line be drawn, cutting the bases, the two parts are equivalent.

438. In every trapezoid the triangle which has for its base one leg, and for its vertex the middle point of the other leg, is equivalent to one-half the trapezoid.

439. If any point within a parallelogram, selected at random, be joined to the four vertices, the sum of the areas of either pair of opposite triangles is equivalent to one-half the parallelogram.

440. The area of a trapezoid is equal to the product of one of its legs and the distance of this leg from the middle point of the other.

441. The triangle whose vertices are the middle points of the sides of a given triangle is equivalent to one-fourth the latter.

442. The parallelogram formed in 101 is equivalent to one-half the quadrilateral.

443. If two parallelograms have two contiguous sides respectively equal, and their included angles supplementary, the parallelograms are equivalent.

444. The lines joining the middle point of either diagonal of a quadrilateral to the opposite vertices, divide the quadrilateral into two equivalent parts.

445. The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

Problems of Computation.

446. Compute the area of a right isosceles triangle if the hypothenuse is 100 rods.

447. Compute the area of a rhombus if the sum of its diagonals is 12 inches and their ratio is $3:5$.

448. Compute the area of a right triangle whose hypothenuse is 13 feet and one of whose legs is 5 feet.

449. Compute the area of an equilateral triangle, one of whose sides is 40 feet.

450. The area of a trapezoid is $3\frac{1}{8}$ acres; the sum of the two parallel sides is 242 yards. Find the perpendicular distance between them.

451. The diagonals of a rhombus are 24 feet and 40 feet respectively. Compute its area.

452. The diagonals of a rhombus are 88 feet and 234 feet respectively. Compute its area, and find length of one of its sides.

453. The area of a rhombus is 354,144 square feet, and one diagonal is 672 feet. Compute the other diagonal and one side.

454. The sides of a right triangle are in the ratio of 3, 4, and 5, and the altitude upon the hypothenuse is 20 yards. Compute the area.

455. Compute the area of a quadrilateral circumscribed about a circle whose radius is 25 feet and the perimeter of the quadrilateral 400 feet.

456. Compute the area of a hexagon having the same length of perimeter and circumscribed about the same circle.

457. The base of a triangle is 75 rods and its altitude 60 rods. Find the perimeter of an equivalent rhombus if its altitude is 45 rods.

458. Upon the diagonal of a rectangle 40 yards by 25 yards an equivalent triangle is constructed. Compute its altitude.

459. Compute the side of a square equivalent to a trapezoid whose bases are 56 feet and 44 feet respectively, and each of whose legs is 10 feet.

460. Find what part of the entire area of a parallelogram will be the area of the triangle formed by drawing a line from one vertex to the middle point of one of the opposite sides.

461. In two similar polygons two homologous sides are 15 feet and 25 respectively. The area of the first polygon is 450 square feet. Compute the area of the other polygon.

462. The base of a triangle is 32 feet and its altitude 20 feet. Compute the area of the triangle formed by drawing a line parallel to the base at a distance of 15 feet from the base.

463. The sides of two equilateral triangles are 20 and 30 yards respectively. Compute the side of an equilateral triangle equivalent to their sum.

464. If the side of one equilateral triangle is equivalent to the altitude of another, what is the ratio of their areas?

465. The radius of a circle is 15 feet, and through a point 9 inches from the centre any chord is drawn. What is the product of the two segments of this chord?

466. A square field contains $5\frac{5}{8}$ acres. Find the length of fence that incloses it.

467. A square field 210 yards long has a path round the *inside* of its perimeter which occupies just one-seventh of the whole field. Compute the width of the path.

468. A street $1\frac{1}{4}$ miles long contains 5 acres. How wide is the street?

469. The perimeter of a rectangle is 72 feet, and its length is twice its breadth. What is its area?

470. A chain 80 feet long incloses a rectangle 15 feet wide. How much more area would it inclose if the figure were a square?

471. The perimeter of a square, and also of a rectangle whose length is four times its breadth, is 400 yards. Compute the difference in their areas.

472. A rectangle whose length is 25^m is equivalent to a square whose side is 15^m. Which has the greater perimeter, and how much?

473. The perimeters of two rectangular lots are 102 yards and 108 yards respectively. The first lot is $\frac{5}{8}$ as wide as it is long, and the second lot is twice as long as it is wide. Compute the difference in the value of the two lots at $1 per square foot.

474. A rhombus and a rectangle have equal bases and equal areas. Compute their perimeters if one side of the rhombus is 15 feet and the altitude of the rectangle is 12 feet.

475. The altitudes of two triangles are equal, and their bases are 20 feet and 30 feet respectively. Compute the base of a triangle equivalent to their sum and having an altitude $\frac{1}{4}$ as great.

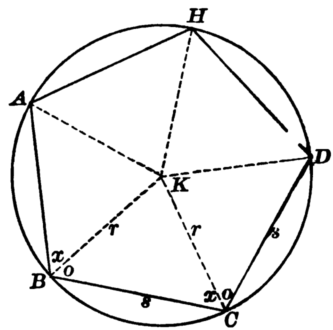
REGULAR POLYGONS AND CIRCLES.

See 328.

476. An equilateral polygon inscribed in a circle is regular.
Sug. See Theorem 193.

477. An equiangular polygon circumscribed about a circle is regular.

478. If a polygon be regular, a circle can be circumscribed about it; i.e. one circumference can be constructed which shall pass through all its vertices.



Post. Let $ABCDH$ be a regular polygon of n sides.

We are to prove that a circumference can be constructed which will pass through all its vertices.

Dem. A circumference may be passed through any three vertices, as A, B, and C. (See Theorem 218.)

From the centre, K, of this circumference draw lines to all the vertices.

$\angle ABC = \angle BCD$. Why?

What relation exists between the two triangles AKB and BKC? Why?

What relation, then, exists between the two angles ABK and BCK? Why?

What relation must consequently exist between the two angles KBC and KCD? Why?

How, then, do the two triangles KBC and KCD compare? Why?

What must therefore be the relation between KD and KC? Why?

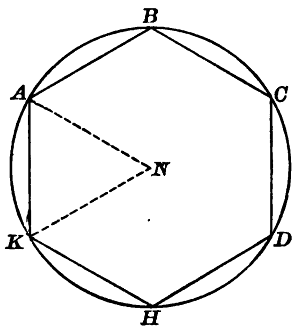
What must be true, then, of the circumference passing through the vertices A, B, and C, as regards the vertex D?

In a similar manner, fix the position of the circumference with regard to each of the other vertices.

479. If a polygon be regular, a circle may be inscribed in it; i.e. a circle may be constructed which shall have the sides of the polygon as tangents.

Sug. First *circumscribe* a circle by Theorem 478, then consult 214.

480. One side of a regular hexagon is equal to the radius of the circumscribed circle.



Post. Let $ABCDHK$ be a regular hexagon, and let N be the centre of the circumscribed circle (Theorem 478), and let NA and NK be drawn.

We are to prove $AK = AN$.

Dem. The arc AK is what part of the entire circumference? Why?

What is the value of the angle N then?

What, then, must be the value of

$$\angle NAK + \angle NKA? \quad \text{Why?}$$

What relation exists between the $\measuredangle$ NAK and NKA? Why?

What, then, must be the value of each one of them?

From this relation of the three angles what must be the relation of the sides of the $\triangle AKN$?

Hence $AK = AN$. Q.E.D.

481. If the circumference of a circle be divided into any number of equal arcs, the chords joining the successive points of division will form a regular polygon.

482. If the circumference of a circle be divided into any number of equal arcs, the tangents at the points of division will form a regular polygon.

483. Tangents to a circumference at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides as the inscribed polygon.

484. *Def.* The radius of the *circumscribed* circle is called the *radius of the polygon.*

The radius of the *inscribed* circle is called the *apothegm of the*

polygon. The common centre of the circumscribed and inscribed circles is also the *centre of the polygon.*

The angle formed by two polygonal radii is called the *polygonal central angle,* and that formed by two sides simply the *polygonal angles.*

485. If all the radii of a regular polygon of n sides be drawn, how many triangles will thus be formed?

What *kind* of triangles will they be?

What will be their relation to one another as regards magnitude?

Find an expression for the *central polygonal angle* in terms of n and *a right angle.*

Find an expression for the *polygonal angle* in terms of the same quantities. (See 329.)

What relation exists between the *polygonal angle* and the *polgonal central angle?* Prove.

$$\frac{2(n-2)}{n} = \text{Pol. angle,} \quad \text{and} \quad \frac{4}{n} = \text{Cent. angle.}$$

Sug. See above and compare the two expressions.

How does the radius of the polygon divide the *polygonal angle?*

486. If radii be drawn to a regular circumscribed polygon, and chords of the circle be drawn joining the successive points of intersection, these chords will form a regular polygon of the same number of sides as the circumscribed polygon, and the sides of the two polygons will be parallel, two and two.

487. If tangents be drawn at the middle points of the several arcs subtended by the sides of a regular polygon, these tangents will form a regular polygon whose vertices lie on the radii extended of the inscribed polygon, and whose sides are respectively parallel to those of the latter.

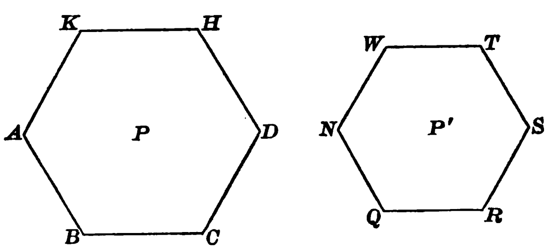
488. If the vertices of a regular inscribed polygon of n sides are joined to the middle points of the arcs subtended by the sides of the polygon, the lines thus drawn will form a regular inscribed polygon of $2n$ sides.

489. If tangents are drawn at the middle points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon of n sides, a regular circumscribed polygon of $2n$ sides will thus be formed.

490. The perimeter of a regular inscribed polygon of n sides is less than the perimeter of the regular polygon of $2n$ sides inscribed in the same circle.

491. The perimeter of the regular circumscribed polygon of n sides is greater than the perimeter of the regular polygon of $2n$ sides circumscribed about the same circle.

492. Two regular polygons of the same number of sides are similar.



Post. Let P and P' be two regular polygons, each having n sides. We are to prove that they are similar; i.e. that their homologous sides form a proportion, and that they are mutually equiangular.

Dem. What is the relation between all the angles of polygon P? Of polygon P'? Why?

I. What is the value of *one* of the polygonal angles, as A, of polygon P? See 329 and 485.

What is the value, expressed in same terms, of one of the polygonal angles, as N, of the polygon P'?

What, then, is the relation between the angles A and N?

Hence, what must be true regarding the corresponding angles of the two polygons?

II. What is the relation between AB and BC? Why?

Hence $$\frac{AB}{BC} = 1.$$

What is the relation between NQ and QR? Why?

Hence $$\frac{NQ}{QR} = 1.$$

Therefore $$\frac{AB}{BC} = \frac{NQ}{QR}.$$ Why?

Or $AB : BC :: NQ : QR$, (proportion form)

or $AB : NQ :: BC : QR$. Why?

The pupil should finish the demonstration.

493. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of any two homologous sides.

Sug. Consult 431.

494. The perimeters of two regular polygons of the same number of sides are in the same ratio as the radii of their circumscribed circles.

Sug. Consult 359 and 492.

495. The perimeters of two regular polygons of the same number of sides are in the same ratio as their apothegms.

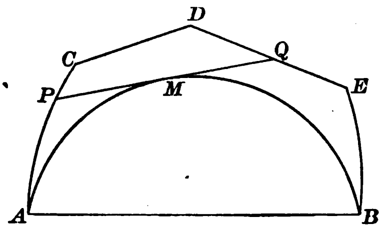
496. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of the radii of their circumscribed circles.

497. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of their apothegms.

498. The area of a regular polygon is equal to one-half the product of its perimeter and apothegm.

Sug. Draw radii, then consult 485.

499. Any curved or polygonal line which envelops an arc of a circle from one extremity to another is longer than the enveloped arc.



Post. Let AMB be the arc of a circle. Then, of all enveloping lines, there must be one shorter than any of the others. Let $ACDEB$ be that line. (Of course, if there should be others equal in length to $ACDEB$, the argument would not be vitiated. The essential condition is, that there shall be *none shorter.*)

Dem. There are three possible relations between the arc AMB and the enveloping line $ACDEB$.

I. $ACDEB = AMB.$

II. $ACDEB < AMB.$

III. $ACDEB > AMB.$

Let us suppose Case I. to be true. Then join any two points in $ACDEB$ by a line that shall not cut the arc AMB, as PQ. (The possibility of constructing this line cannot be challenged, as $ACDEB$ *envelops* the arc AMB.)

Then, since $PQ < PC + CD + DQ$, $APQED > ACDEB$. But $APQED$ is a line that envelops the arc AMB. Hence we have arrived at the result that this enveloping line is shorter than $ACDEB$. This is in direct conflict with the

hypothesis, which was that the latter was the shortest enveloping line. Hence Case I. is inadmissible. Case II. leads to the same result. Hence Case III. is alone true.

In the same manner it can be proved that any convex line which returns into itself is shorter than any line enveloping it on all sides, whether the enveloping line touches the given convex line in one or several places or surrounds without touching it.

500. The circumference of a circle is greater than the perimeter of any polygon inscribed in it.

501. The circumference of a circle is less than the perimeter of any polygon circumscribed about it.

Sug. Consult 499.

502. The area of a regular inscribed polygon of $2n$ sides is greater than the area of a regular polygon of n sides inscribed in the same circle.

503. The area of a regular circumscribed polygon of $2n$ sides is less than the area of a regular polygon of n sides circumscribed about the same circle.

504. If the number of sides of a regular inscribed polygon be continuously doubled, its perimeter will as continuously increase in length, and consequently approach nearer and nearer the length of the circumference.

505. If the number of sides of a regular inscribed polygon be continuously doubled, its apothegm will as continuously increase in length, and consequently approach nearer and nearer the length of the radius.

506. If the number of sides of a regular circumscribed polygon be continuously doubled, its perimeter will as continuously decrease in length, and consequently approach nearer and nearer the length of the circumference.

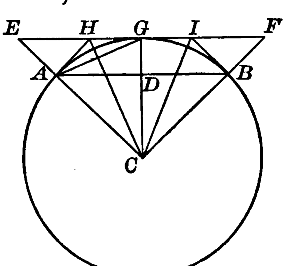
507. If the number of sides of a regular circumscribed polygon be continuously doubled, its radius will as continuously decrease in length, and consequently approach nearer and nearer the length of the radius of the circle.

508. If the number of sides of a regular inscribed polygon be continuously doubled, its area will as continuously increase, and consequently approach nearer and nearer the area of the circle.

509. If the number of sides of a regular circumscribed polygon be continuously doubled, its area will as continuously decrease, and consequently approach nearer and nearer the area of the circle.

510. If, in 504 to 509 inclusive, the number of sides be doubled an infinite number of times, the approach will be infinitely near; i.e. they will coincide: hence *a circle may be regarded as a regular polygon with an infinite number of sides,* with the circumference its perimeter, and radius its apothegm.

511. The area of a regular inscribed polygon of $2n$ sides is a mean proportional between the areas of two polygons, each of n sides, one inscribed within, and the other circumscribed about, the same circle.



Post. Let AB be one side of the regular inscribed polygon of n sides, and EF one side of the regular circumscribed polygon of n sides and parallel to AB. Let C be the centre of the circle. Draw the radii CA, CG, and CB, G being the point of contact of the tangent EF. Then CA and CB, if extended, will pass through points E and F respectively. At A and B construct the tangents AH and BI, and join CH. Then HI will be

one side of the regular circumscribed polygon of $2n$ sides. Designate the area of this polygon by P'.

Again, let p represent the area of the inscribed polygon whose side is AB, or the polygon of n sides; P the area of the circumscribed polygon of n sides, or whose side is EF; and p' the area of the inscribed polygon of $2n$ sides, and whose side is AG.

We are to prove $p : p' :: p' : P$.

Dem. It is evident that the areas of the ▵ ACD, ACG, and ECG are $\frac{1}{2n}$ part of the respective polygons p, p', and P.

$$\text{Area } \triangle ACD : \text{Area } \triangle ACG :: CD : CG;$$

and $$\text{Area } \triangle ACD : \text{Area } \triangle ACG :: p : p'.$$

$$\therefore p : p' :: CD : CG.$$

Again, $$\text{Area } \triangle CAG : \text{Area } \triangle CEG :: CA : CE,$$

and $$\text{Area } \triangle CAG : \text{Area } \triangle CEG :: p' : P.$$

$$\therefore p' : P :: CA : CE.$$

But, since AD is parallel to EG, the two ▵ CAD and CGE are similar.

Hence $$CD : CG :: CA : CE.$$

$$\therefore p : p' :: p' : P.$$ Q.E.D.

512. The area of a regular circumscribed polygon of $2n$ sides is equal to the quotient obtained by dividing twice the product of the areas of two regular polygons, each of n sides, one inscribed within, and the other circumscribed about, the same circle, by the sum of the areas of two regular polygons, one of n and the other of $2n$ sides, both inscribed in the same circle. (See diagram and postulate of previous theorem.)

We are to prove $P' = \frac{2pP}{p + p'}$.

Dem. Since CH bisects the $\angle ECG$,

$$GH : HE :: CG : CE. \quad \text{(See Theorem 323.)}$$

But $\quad \text{Area } \triangle CGH : \text{Area } \triangle CHE :: GH : HE.$

$$\therefore \text{Area } \triangle CGH : \text{Area } \triangle CHE :: CG : CE.$$

Again, $CG : CE :: CD : CA$, or $CG : CE :: CD : CG$.

But $$p : p' :: CD : CG. \quad \text{(Theorem 511.)}$$

$$\therefore p : p' :: CG : CE.$$

$$\therefore \text{Area } \triangle CGH : \text{Area } \triangle CHE :: p : p'.$$

Hence

$$\text{Area } \triangle CGH : \text{Area } \triangle CGH + \text{Area } \triangle CHE :: p : p + p',$$

or $$\text{Area } \triangle CGH : \text{Area } \triangle CGE :: p : p + p',$$

or $$2 \text{ Area } \triangle CGH : \text{Area } \triangle CGE :: 2p : p + p',$$

or $$\text{Area } \triangle CHI : \text{Area } \triangle CGE :: 2p : p + p'.$$

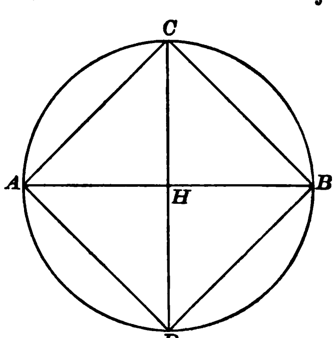
But areas of $\triangle$s CHI and CGE are $\frac{1}{2n}$ part respectively of the polygons P' and P.

Hence $\quad \text{Area } \triangle CHI : \text{Area } \triangle CGE :: P' : P.$

$$\therefore P' : P :: 2p : p + p'.$$

Whence $$P' = \frac{2pP}{p+p'}. \qquad \text{Q.E.D.}$$

513. The chords which join the extremities of two perpendicular diameters form a square.



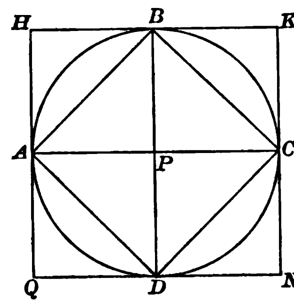
Post. Let CB, BD, DA, and AC be chords of the circle $ACBD$ connecting the extremities of the two perpendicular diameters CD and AB.

We are to prove that $ACBD$ is a square.

Sug. Consult 487.

514. The diagonals of an inscribed square will be diameters of the circumscribing circle.

515. The tangents to a circle whose points of contact are the vertices of an inscribed square will form a square.



Post. Let $ABCD$ be a square inscribed in the circle whose centre is P; BD and AC its diagonals; and HK, KN, NQ, and QH, tangents whose points of contact are the vertices of the inscribed square.

We are to prove that $HKNQ$ is a square.

Sug. Consult 482.

516. The area of an inscribed square is equivalent to twice the square of the radius. (Use diagram of previous theorem.)

It is evident that the area of the right triangle DPC is one-half that of the square upon PC, or the radius of the circle.

The area of the square $ABCD$ is equal to four times the area of the triangle DPC,

or $$\text{Area } \triangle DPC = \frac{r^2}{2} \text{ (representing radius by } r\text{)},$$

and $$4 \times \text{Area } \triangle DPC = 2r^2 = \text{Area } ABCD. \qquad \text{Q.E.D.}$$

517. The area of a circumscribed square is equal to four times the square of the radius of the circle. (Use same diagram as before.)

It can easily be shown that the square $HKNQ$ is composed of four smaller squares, each of which is the square of the radius; hence

$$\text{Area } HKNQ = 4r^2. \qquad \text{Q.E.D.}$$

518. *Problem.* From 511 and 512 we have

$$\text{I. } p' = \sqrt{p \times P} \quad \text{and} \quad \text{II. } P' = \frac{2pP}{p + p'}.$$

Letting p and P represent inscribed and circumscribed squares respectively, we have from 516 and 517,

$$p = 2r^2 \quad \text{and} \quad P = 4r^2.$$

By substituting these values in I., we have

$$p' = 2.82843r^2.$$

Then by substituting values of p, p', and P in II., we have

$$P' = 3.31371r^2.$$

Thus we have computed the areas of the regular inscribed and circumscribed octagons in terms of the radius of the circle.

Again, calling these p and P, p' and P' will be polygons of sixteen sides; and using the formulæ as before, the areas of the latter may be computed. Repeating the process, those of thirty-two, sixty-four, etc., sides may be, in like manner, computed.

Below are the tabulated results to seven decimal places for thirteen doublings of the number of sides of the polygons.

No. Sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	$2.0000000r^2$	$4.0000000r^2$
8	$2.8284271r^2$	$3.3137085r^2$
16	$3.0614675r^2$	$3.1825979r^2$
32	$3.1214452r^2$	$3.1517240r^2$
64	$3.1365485r^2$	$3.1441184r^2$
128	$3.1403312r^2$	$3.1422236r^2$
256	$3.1412773r^2$	$3.1417504r^2$
512	$3.1415138r^2$	$3.1416321r^2$
1024	$3.1415729r^2$	$3.1416025r^2$
2048	$3.1415877r^2$	$3.1415951r^2$
4096	$3.1415914r^2$	$3.1415933r^2$
8192	$3.1415923r^2$	$3.1415928r^2$
16384	$3.1415925r^2$	$3.1415927r^2$
32768	$3.1415926r^2$	$3.1415926r^2$

Hence, since the area of the circle is greater than that of the inscribed polygon and less than that of the circumscribed polygon, $3.1415926r^2$ must be the area of the circle correct to within less than one ten-millionth part of r^2. But by continuing the process, the areas of the two polygons may be made to agree to any desired number of decimal places, and therefore such result may be taken as the area of the circle without sensible error. If r be taken as unity, it would, of course, vanish from the expression, and consequently 3.1415926 may be taken as *the area of a circle whose radius is unity.*

519. If, in 511, we let P, P', p, and p' represent perimeters instead of areas, then,

$$\text{I. } P' = \frac{2pP}{p+P}, \text{ and II. } p' = \sqrt{p \times P'}.$$

Dem. $\quad P : p :: EC : AC$ (or CG),

$$EH : HG :: EC : CG.$$

$$\therefore \quad P : p :: EH : HG.$$

$$\therefore \quad P + p : 2p :: EH + HG : 2\,HG.$$

$$\therefore \quad P + p : 2p :: EG : HI.$$

But $\quad P : P' :: EG : HI.$

$$\therefore \quad P + p : 2p :: P : P'.$$

$$\therefore \quad P' = \frac{2pP}{p+P}. \qquad \text{I.}$$

Again, $\quad AD : AG :: p : p'$,

and $\quad AG : HI :: p' : P'.$

$$\therefore \quad \frac{AG}{2} : \frac{HI}{2} :: p' : P'.$$

But $\quad \frac{AG}{2} : \frac{HI}{2} :: AD : AG.$

$$\therefore \quad p : p' :: p' : P'.$$

$$\therefore \quad p' = \sqrt{p \times P'}. \qquad \text{II.}$$

We will now use these formulæ for the computation of perimeters in a manner similar to that in 518, beginning with the circumscribed and inscribed squares.

The perimeter of the circumscribed square in terms of the diameter, calling the latter D, is, obviously, $4D$. In the right triangle APD (515),

$$\overline{AD}^2 = \overline{AP}^2 + \overline{PD}^2 = 2\,\overline{AP}^2 = 2r^2 = 2\left(\frac{D}{2}\right)^2 = \frac{D^2}{2}.$$

$$\therefore\ AD = \frac{D}{\sqrt{2}},\quad 4\,AD = \frac{4}{\sqrt{2}}D = D\sqrt{8}.$$

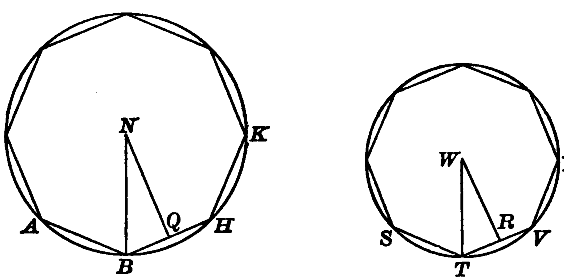
Hence the perimeter of the inscribed square is $2.8284271\,D$. Continuing as in 518 and tabulating the results, we have the following:

No. Sides.	Perimeter of Inscribed Polygon.	Perimeter of Circumscribed Polygon.
4	$2.8284271\,D$	$4.0000000\,D$
8	$3.0614675\,D$	$3.3137085\,D$
16	$3.1214452\,D$	$3.1825979\,D$
32	$3.1365485\,D$	$3.1517249\,D$
64	$3.1403312\,D$	$3.1441184\,D$
128	$3.1412773\,D$	$3.1422236\,D$
256	$3.1415138\,D$	$3.1417504\,D$
512	$3.1415729\,D$	$3.1416321\,D$
1024	$3.1415877\,D$	$3.1416025\,D$
2048	$3.1415914\,D$	$3.1415951\,D$
4096	$3.1415923\,D$	$3.1415933\,D$
8192	$3.1415924\,D$	$3.1415928\,D$
16384	$3.1415925\,D$	$3.1415927\,D$
32768	$3.1415926\,D$	$3.1415926\,D$

Hence, since the circumference of the circle is greater than the perimeter of the *inscribed* polygon, and less than that of

the *circumscribed* polygon, $3.1415926D$ must be the circumference of the circle correct to within less than one ten-millionth part of D. But by continuing the process the perimeters of the two polygons may be made to agree to any desired number of decimal places, and therefore such result may be taken as the circumference of the circle without sensible error. If D be taken as unity, it would, of course, vanish from the expression, and consequently 3.1415926 may be taken as *the circumference of a circle whose diameter is unity.*

520. The circumferences of two circles are in the same ratio as their radii, and also their diameters.



Post. Let $ABHK$, etc., and $STVY$, etc., be two circles whose centres are N and W, and designate the circumferences by C and c, and their radii by R and r, and their diameters by D and d, the capital letter referring to the larger circle.

We are to prove

I. $C:c::R:r$.

II. $C:c::D:d$.

Cons. Inscribe in each a regular polygon of n sides, and construct the radii NB and WT, and the apothegms NQ and WR.

Dem. Designating the perimeters by P and p,

$P:p::NQ:WR$. Why?

If now we inscribe polygons with double the number of sides, and continue this process indefinitely, the perimeters will coincide with the circumferences, and the apothegms with the radii.

Hence I. $C : c :: R : r,$

and $C : c :: 2R : 2r;$ Why?

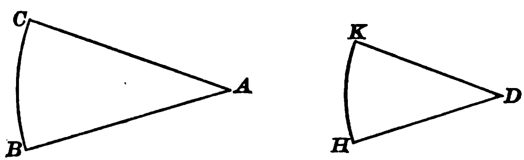
or II. $C : c :: D : d.$ Q.E.D.

521. The areas of two circles are in the same ratio as the squares of their radii and of their diameters.

Sug. Use method similar to the above, and consult 496 and 497.

521 (*a*). *Def.* Similar *arcs, sectors,* and *segments* are those that correspond to equal central angles.

522. Similar arcs are in the same ratio as the radii of the circumferences of which they are a part, and also as the diameters.



Post. Let CB and KH be two similar arcs, and A and D the centres of the circumferences of which they are a part.

Cons. Draw the radii AC, AB, DK, and DH.

(Designate *circumferences, diameters,* and *radii,* as before.)

We are to prove

I. Arc CB : Arc $KH :: R : r.$

II. Arc CB : Arc $KH :: D : d.$

Dem. $\angle A = \angle D.$ Why?

Hence arc CB is the same part of the circumference C, as arc KH is of the circumference c.

$\therefore$ Arc CB : Arc $KH :: C : c.$

But $R : r :: C : c.$ (See Theorem 520.)

$\therefore$ I. $\text{Arc } CB : \text{Arc } KH :: R : r;$ Why?

II. $\text{Arc } CB : \text{Arc } KH :: D : d.$ Why?

Q.E.D.

523. The areas of similar sectors are in the same ratio as the squares of the radii, and also of the diameters, of the circles of which they are a part.

524. The area of a circle is equal to one-half the product of its circumference and radius.

Sug. Consult 498 and 510.

525. The area of a sector is equal to one-half the product of its arc and radius.

526. The areas of similar segments are in the same ratio as the squares of their radii, the squares of their diameters, and as the squares of their chords.

527. Let us designate the circumference of a circle whose diameter is unity by π, and the circumference of any other circle by C; its diameter by D; its radius by R; and its area by A.

Then $C : \pi :: D : 1.$ Why?

Hence I. $C = \pi D;$

whence II. $\frac{C}{D} = \pi,$

or III. $C = \pi \times 2R.$

Multiplying both members of this equation by $\frac{R}{2}$, we have

$$\frac{CR}{2} = \pi R^2.$$

But, by 524, $\frac{CR}{2}$ = the area of the circle; hence

$$\text{IV.} \quad A = \pi R^2.$$

528. Hence *the area of any circle is equal to the square of its radius multiplied by the constant quantity* π, *and the circumference of every circle is equal to the product of its diameter* (*or twice its radius*) *by the same quantity* π.

From II. above, it is readily seen that π is the ratio of the circumference of any circle to its diameter, or of a semi-circumference to its radius.

The exact numerical value of π can be only approximately expressed. As computed in 352 it is 3.1415926, but for practical purposes in computing its value is usually taken as 3.1416.

The symbol π is the first letter of the Greek word meaning perimeter or circumference.

529. The *quadrature of the circle* is the problem which requires the finding of a square which shall be equal in area to that of a circle with a given radius. Now since the area of a circle is equal to its circumference multiplied by one-half its radius, if a straight line of same length as circumference be taken as the base of a rectangle, and one-half the radius as its altitude, their product will be the area of the rectangle, also of the circle. It is also evident that, if a line which is a mean proportional between these two be taken as the side of a square, the area of this square will be equal to that of the rectangle, and consequently to that of the circle. It will be shown in the series of problems of construction (Prob. 736) how this mean proportional can be found; hence, to "square the circle" we must be able to find the circumference when the radius is known, or *vice versa*. For accomplishing this, we must know the ratio of the circumference to its diameter or radius. But this ratio, as has been remarked before, can be only approximately expressed; for, as the higher mathematics prove, the circumference and diameter are incommensurable, but the approximation has been carried so far that the error is infinitesimal. Archimedes, about 250 B.C., was the first to assign an approximate value to π. He found that it must be

between 3.1428 and 3.1408. In 1640 Metius computed it correctly to 6 places. Later, in 1579, Vieta carried the approximation to 10 places, Van Ceulen to 36 places, Sharp to 72 places, Machin to 100 places, De Lagny to 128 places, Rutherford to 208 places, and Dr. Clausen to 250 places. In 1853 Rutherford carried it to 440 places, and in 1873 Shanks computed it to 707 places, but these latter results do not appear to have been verified. The following is its value to 208 places, as computed by Rutherford:

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534271706798214808651328230664709384460955058223172535940812848473781392038633830215747399600825931259129401832806517 44+.$$

530. Some idea of the accuracy of the above value may be formed from the following statement taken from Peacock's Calculus: "If the diameter of the universe be 100,000,000,000 times the distance of the sun from the earth (about 93,000,000 miles), and if a distance which is 100,000,000,000 times this diameter be divided into parts, each of which is one 100,000,000,000th part of an inch; then if a circle be described whose diameter is 100,000,000,000 times that distance, repeated 100,000,000,000 times as often as each of those parts of an inch is contained in it; then the error in the circumference of this circle, as computed from this approximation, will be less than one 100,000,000,000th part of the one 100,000,000,000th part of an inch."

ADVANCE THEOREMS.

531. In two circles of different radii, angles at the centres subtending arcs of equal length are to each other inversely as the radii.

532. If, from any point within a regular polygon of n sides, perpendiculars be drawn to all the sides, the sum of these perpendiculars is equal to n times the apothegm.

533. If perpendiculars be drawn from the vertices of a regular polygon to any diameter of the circumscribed circle, the sum of the perpendiculars upon one side of the diameter is equal to the sum of those on the other side.

534. An equiangular polygon inscribed in a circle is regular if the number of its sides be odd.

535. An equilateral polygon circumscribed about a circle is regular if the number of its sides be odd.

536. The sum of the squares of the lines joining any point in the circumference of a circle with the vertices of an inscribed square is equal to twice the square of the diameter of the circle.

537. The area of the ring included between two concentric circles is equal to the area of the circle whose diameter is that chord of the outer circle which is tangent to the inner.

538. If the sides of a right triangle be the homologous sides of similar polygons, the area of the polygon on the hypothenuse is equal to the sum of the areas of the other two.

539. If three circles be described upon the sides of a right triangle as diameters, the area of that described upon the hypothenuse is equal to the sum of the areas of the other two.

540. If upon the legs of a right triangle semi-circumferences are described outwardly, the sum of the areas contained between these semi-circumferences and the semi-circumference passing through the three vertices is equal to the area of the triangle.

541. If the diameter of a circle be divided into two parts, and upon these parts semi-circumferences are described on

opposite sides of the diameter, these semi-circumferences will divide the circle into two parts which have the same ratio as the two parts of the diameter.

542. If two chords of a circle be perpendicular to each other, the sum of the areas of the four circles described upon the four segments as diameters will be equal to the area of the given circle.

543. If squares be constructed outwardly upon the sides of a regular hexagon, their exterior vertices will be the vertices of a regular dodecagon.

544. The radius of a regular inscribed polygon is a mean proportional between its apothegm and the radius of a regular polygon of double the number of sides circumscribed about the same circle.

545. The area of a regular dodecagon is equal to three times the square of its radius.

546. If the radius of a circle be divided in extreme and mean ratio, the larger part will be the side of a regular decagon inscribed in the same circle.

547. The apothegm of a regular inscribed pentagon is equal to one-half the sum of the radius of the circle and the side of the regular decagon inscribed in the same circle.

548. The square of the side of a regular inscribed pentagon is equivalent to the sum of the squares of the side of the regular inscribed decagon and the radius of the circle.

Problems of Computation.

549. Compute the side of a regular inscribed trigon in terms of the regular trigon circumscribed about the same circle. Compare their areas.

550. Compute the side of an inscribed square in terms of the square circumscribed about the same circle.

551. Compute the apothegm of a regular inscribed trigon in terms of the regular hexagon inscribed in the same circle.

552. Compute the apothegm of a regular inscribed hexagon in terms of the regular trigon inscribed in the same circle.

553. Regular trigons and hexagons are both inscribed and circumscribed about the same circle. Compare their areas.

554. Compute the area of a regular polygon of 24 sides inscribed in a circle whose radius is 10 inches.

555. Compute the perimeter of a regular pentagon inscribed in a circle whose radius is 12 feet.

556. The perimeter of a regular hexagon is 480 feet, and that of a regular octagon is the same. Which has the greater area, and how much?

557. If paving blocks are in the shape of regular polygons (i.e. their cross-sections), how many shapes can be employed in order to completely fill the space?

558. Compute the diameter of a circle whose circumference is 12 feet and 10 inches.

559. The diameter of a carriage wheel is 4 feet and 3 inches. How many revolutions does it make in traversing one-fourth of a mile?

560. What is the width of the ring between two concentric circumferences whose lengths are 480 feet and 360 feet?

561. Find the length of an arc of 36° in a circle whose diameter is 36 inches.

562. In raising water from the bottom of a well by means of a wheel and axle, it was found that the axle, whose diameter was 8 inches, made 20 revolutions in raising the bucket. Compute the depth of the well.

563. Find the central angle subtending an arc 6 feet and 4 inches long, if the radius of the circle be 8 feet and 2 inches.

564. If the radius of a circle is 5 feet and 3 inches, find the perimeter of a sector whose angle is 45°.

565. If the central angle subtending an arc 10 feet and 6 inches long is 72°, what is the length of the radius of the circle?

566. If the length of a meridian of the earth be 40,000,000 metres, what is the length of an arc of 1″?

567. Two arcs have the same angular measure, but the length of one is twice that of the other. Compare the radii of those arcs.

568. Compute the area of a circle whose circumference is 100 yards.

569. Two arcs have the same length, but their angular measurements are 20° and 30° respectively. If the radius of the first arc is 6 feet, compute the radius of the other.

570. Find the circumference of a circle whose area is 2 acres and 176 square yards.

571. The diameter of a circle is 40 feet. Find the side of a square which is double the area of the circle.

572. The area of a square is 196 square rods. Find the area of the circle inscribed in the square.

573. A circular fish-pond which covers an area of 5 acres and 100 square rods is surrounded by a walk 5 yards wide. Compute the cost of gravelling the walk at $6\frac{1}{4}$ cents per square yard.

574. What must be the width of a walk around a circular garden containing $1\frac{3}{4}$ acres, in order that the walk may contain exactly one-fourth of an acre?

575. A carpenter has a rectangular piece of board 15 inches wide and 20 inches long, from which he wishes to cut the largest possible circle. How many square inches of the board must he cut away?

576. The perimeters of a circle, a square, and a regular trigon, are each equal to 144 feet. Compare their areas.

577. If the radius of a circle be 12 inches, what is the radius of a circle 10 times as large?

578. What will it cost to pave a circular court 30 feet in diameter, at 54 cents per square foot, leaving in the centre a hexagonal space, each side of which measures 4 feet?

579. A circle 18 feet in diameter is divided into three equivalent parts by two concentric circumferences. Find the radii of these circumferences.

580. If the chord of an arc be 720 feet, and the chord of its half be 369 feet, compute the diameter of the circle.

581. The chord of half an arc is 17 feet, and the height of the arc 7 feet. Compute the diameter of the circle.

582. The radius of a circle is 12 feet; the chords which subtend two contiguous arcs are 6 feet and 9 feet respectively. Compute the chord subtending the arc equal to the sum of the other two.

583. The lengths of two chords, drawn from the same point in the circumference of a circle to the extremities of a diameter, are 6 feet and 8 feet respectively. Compute the area of the circle.

584. The chord of an arc is 32 inches, and the radius of the circle is 34 inches. Compute the length of the arc.

585. The diameter of a circle is 106 feet. Compute the lengths of the two arcs into which a chord 90 feet long divides the circumference.

586. The area of a sector is 385 square feet, and the angle of the sector is 36°. Compute the radius of the circle and perimeter of the sector.

587. Compute the area of a segment, if the chord of the arc is 56 feet and the radius of the circle is 35 feet.

588. A room 20 feet long and 15 feet wide has a recess at one end in the shape of the segment of a circle, the chord being 15 feet, and its greatest width 4 feet. Compute the area of the entire room.

589. Compute the number of square feet of brick that would be required in blocking up one of the arches of a railway viaduct, if the span of the arch is 60 feet, height above the piers 20 feet, and distance from the ground to the spring of the arch 20 feet.

590. Compute the area of a circle in which the chord, 3 feet long, subtends an arc of 120°.

591. Compute the area of a segment whose arc is 300°, the radius of the circle being 20 inches.

592. The areas of two concentric circles are as 5 to 8. The area of that part of the ring which is contained between two radii making the angle 45° is 300 square feet. Compute the radii of the two circles.

593. What is the altitude of a rectangle equivalent to a sector whose radius is 15 feet, if the base of the rectangle is equal to the arc of the sector?

594. Compute the radius of a circle, if its area is doubled by increasing its radius one foot.

595. The radius of a circle is 10 inches. Through a point exterior to the circle two tangents are drawn, making an angle of 60°. Compute the area of the figure bounded by the tangents and the intercepted arc.

596. Three equal circles are drawn tangent to each other, with a radius of 12 feet. Compute the area contained between the circles.

597. Upon each side of a square, as a diameter, a semicircumference is described within the square. If the side of the square is 10 inches, find the sum of the areas of the four leaves.

598. In a circle whose radius is 100 feet two parallel chords are drawn on the same side of the centre, one equal to the side of a regular hexagon, and the other to the side of a regular trigon, both inscribed in the given circle. Compute the area of the circle comprised between the two parallel chords.

MAXIMA AND MINIMA.

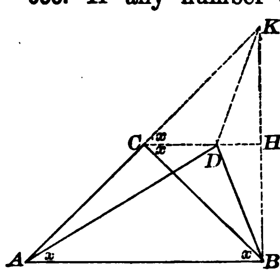
599. *Def.* Among quantities of the same kind, that which is greatest is called the *maximum* (plural, *maxima*), and that which is the smallest is called the *minimum* (plural, *minima*).

For example: of all lines inscribed in the same circle, the diameter is the greatest, and therefore is a *maximum;* and of all lines drawn from the same point to a given straight line, that which is perpendicular is the shortest, and is therefore a *minimum.*

Def. If two or more plane figures have equal perimeters, they are said to be *isoperimetric.*

Def. A plane figure is said to be a *maximum* or a *minimum* when its *area* is a *maximum* or a *minimum.*

600. If any number of triangles have the same or equal bases and equal areas, that which is isosceles has the *minimum perimeter.*



Post. Let ABC and ADB be two triangles having the same base AB and equal areas, and let ACB be isosceles, having CA equal to CB.

We are to prove

$$AC + CB + AB < AD + DB + AB.$$

Or, since $AB = AB$,

we are to prove

$$AC + CB < AD + DB.$$

Cons. From B construct a perpendicular to AB, and extend it to meet AC extended in K. Join DK and draw CH through point D.

Dem. How must the altitudes of the two ▵ ACB and ADB compare?

What must be the position, then, of the line CH relative to AB?

How, then, do the two ∠ HCB and CBA compare? Why? The two ∠ KCH and CAB? Why?

How, then, must the two ∠ KCH and HCB compare? Why?

What is the position of CH relative to KB? Why?

How, then, must CK and CB compare? DK and DB?

Now compare $AC + CK$ with $AD + DK$, and the pupil should be able to write, or give orally, a complete and accurate demonstration of this theorem.

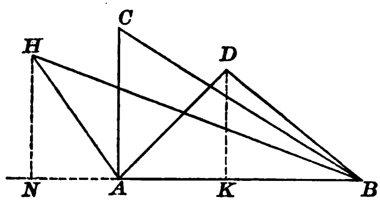
601. If any number of triangles have the same area, that which is equilateral has the minimum perimeter.

602. If any number of triangles have the same or equal bases and equal perimeters, that which is isosceles is the maximum.

Sug. Through the vertex of the isosceles triangle draw a line parallel to the base. Then prove that the vertex of the other triangle cannot fall on that line. Then compare their altitudes, and consequently their areas.

603. If any number of triangles be isoperimetric, that which is equilateral is the maximum.

604. If any number of triangles have two sides in each respectively equal, that in which these sides are perpendicular to each other is the maximum.



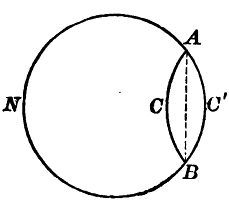
Sug. Place them so that one set of equal sides shall coincide, as AB. Then compare their altitudes DK, CA, and HN, and consequently their areas.

605. If any number of equivalent parallelograms have the same or equal bases, the perimeter of that which is rectangular is the minimum.

606. Of all rectangles of given area, the perimeter of the square is a minimum.

607. If any number of triangles have the same or equal bases and the same or equal altitudes, the perimeter of that which is isoceles is a minimum.

608. Every closed plane figure of given perimeter whose area is a maximum, must be *convex*.



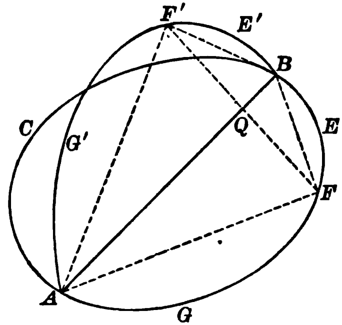
Post. Let $ACBN$ be a plane concave figure, with the straight line AB which joins two of its points in its perimeter lying without.

We are to prove that $ACBN$ cannot be a maximum.

Dem. Conceive the figure CAB to be revolved about AB as an axis till it comes to the position $AC'B$. Then the figures $ACBN$ and $AC'BN$ have equal perimeters, but the area of the latter will exceed that of the former. Hence $ACBN$ cannot be a maximum among isoperimetrical figures. But $ACBN$ is any concave, i.e. non-convex, plane figure. Therefore, of all isoperimetrical plane figures, the maximum must be convex.

609. Of all plane figures that are isoperimetric, that which is a circle is the maximum.

Dem. I. It is evident that, with a given perimeter, an indefinite number of figures of different shapes and areas may be constructed. It is also evident that we can *diminish* the area indefinitely, but cannot thus *increase* it. Consequently, there must be among all these figures having the same perimeter either one maximum figure, or several maximum figures of different forms and equal areas.



II. Let $ACBFG$ be a maximum figure with a given perimeter; then by 608 it must be convex. Let also the line AB divide the perimeter into halves; then it must also divide the area into halves. For suppose one of the parts, as AFB, to be greater than the other, and conceive this part to be revolved on AB as an axis until it comes into the same plane with ACB, and let $AF'B$ be its position after revolution. Hence the perimeter of the figure $AF'BEGA$ is equal to that of the figure $ACBFGA$, but the area of the former is greater than that of the latter. Therefore the figure $ACBFG$ cannot be a maximum. But by hypothesis it is a maximum. Hence AB must bisect the area of $ACBFG$.

Since $ACBFG$ is a maximum, and AB bisects the area, it follows that the figure $AF'BFG$ is also a maximum.

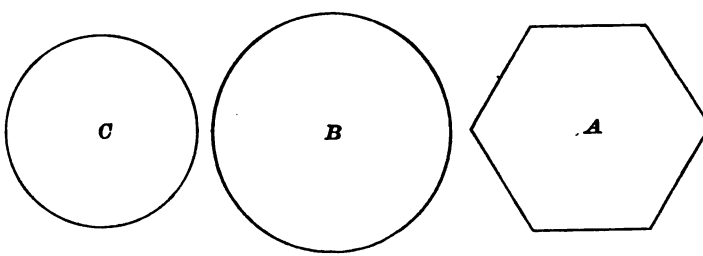
Again, let F be any point in $BEGA$ selected at random, and F' its position after revolution. Join FF', FB, FA, $F'B$, and $F'A$. Then $AF = AF'$, and $FQ = F'Q$. Hence the two triangles AQF and AQF' are equal.

Therefore FF' is perpendicular to AB.

Similarly the two triangles $AF'B$ and AFB are equal.

The triangle AFB must be a maximum, otherwise its area could be increased without increasing its perimeter; i.e. without increasing the lengths of the two chords AF and FB, which would consequently leave the areas of the two segments AGF and FEB unchanged, and therefore make up an area greater than $ABEG$, by which it is evident that $ACBFG$ could not be a maximum; but this also conflicts with the hypothesis which grants that $ACBFG$ is a maximum. Consequently the triangle AFB must be a maximum, and therefore the angle AFB must be a right angle. (See Theorem 604.) But F is any point in the curve $BEFGA$. Hence BFA must be a semicircle, as also ACB. Hence the whole figure $ACBFG$ must be a circle. Q.E.D.

610. Of all plane figures having equal areas, the perimeter of that which is a circle is the minimum.



Post. Let C be a circle, and A any other plane figure having the same area.

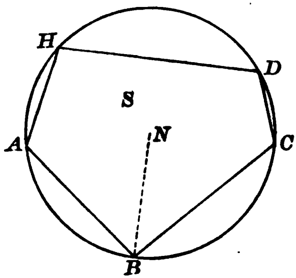
We are to prove Perimeter $C <$ Perimeter A.

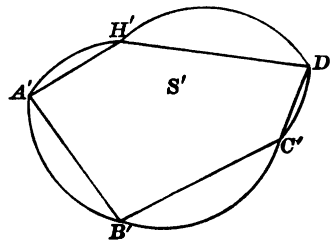
Dem. For let B be a circle having a perimeter equal to that of A. Hence by 609 area B is greater than area A, and hence greater than area C. Hence if area C is less than area B, what must be true of their perimeters, i.e. their circumferences? But by construction

Perimeter $B =$ Perimeter A.

Hence perimeter of C is a minimum. Q.E.D.

611. Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.





Post. Let $ABCDH = P$ and $A'B'C'D'H' = P'$ be two mutually equilateral polygons, having AB equal to $A'B'$, BC equal to $B'C'$, etc., of which $ABCDH$ can be inscribed in a circle, and let N be the centre of such circle.

Cons. With the radius NB construct the arcs $A'B'$, $B'C'$, $C'D'$, etc.

Dem. The Arc $AB =$ Arc $A'B'$, and Arc $BC =$ Arc $B'C'$, etc.

Hence Circumference $ABCDH =$ sum of the Arcs $A'B'$, $B'C'$, etc.

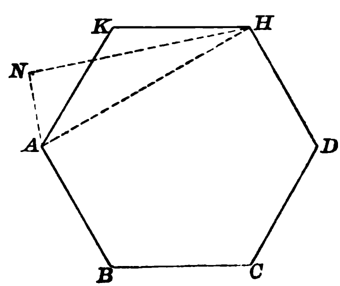
Hence Perimeter of $S =$ Perimeter of S'.

Therefore Area $S >$ Area S'. (Theorem 609.)

But the corresponding segments are equal. Hence, subtracting their respective sums from the above inequality leaves

Area $P >$ Area P'. Q.E.D.

612. Of all isoperimetric polygons of the same number of sides, that which is equilateral is the maximum.



Post. Let $ABCDHK$ be the maximum of all isoperimetrical polygons of any given number of sides.

We are to prove that it is equilateral; i.e. that

$$KH = HD = DC, \text{ etc.}$$

Cons. Connect any two alternate vertices, as AH.

Dem. The $\triangle AKH$ must be a *maximum* of all isoperimetrical triangles having the common base AH; otherwise another triangle, as ANH, could be constructed, having the same perimeter and a *greater area*, in which case the area of the polygon $ABCDHN$ would be greater than that of $ABCDHK$. Hence the latter would not be a maximum. This result conflicts with our hypothesis, which grants that it is a maximum. Therefore the triangle AKH must be a maximum, and consequently *isosceles*. (See Theorem 602.)

Hence $$AK = KH.$$

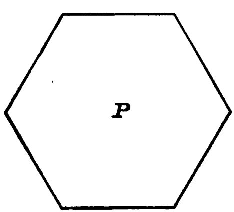
Similarly, by joining KD,

$$KH = HD, \text{ etc.}$$

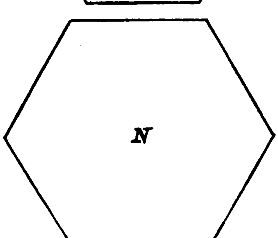
Hence the polygon is equilateral. Q.E.D.

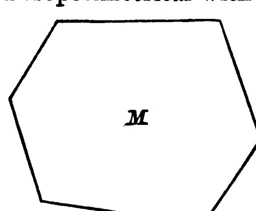
613. The maximum of all equilateral polygons of the same number of sides is that which is regular.

614. Of all polygons having the same number of sides and equal areas, the perimeter of that one which is regular is a minimum.



Post. Let P be a regular polygon, and M any irregular polygon having the same number of sides and same area as P; and let N be a regular polygon having the same number of sides and isoperimetrical with M.





Dem. Area $M < N$. (See 612 and 613.)

Area $M = $ Area P. Why?

$\therefore$ Area $P < N$.

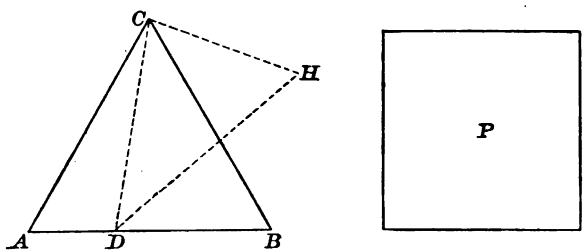
But of two regular polygons of the same number of sides, that which has the less area must have the less perimeter. Why?

Hence Perimeter $P <$ Perimeter N,

and $\therefore$ Perimeter $P <$ Perimeter M.

Hence perimeter of P is a minimum. Q.E.D.

615. Of all isoperimetric regular polygons, that which has the greatest number of sides is the maximum.



Post. Let ABC be a regular *trigon*, and P a regular *tetragon*, having *equal* perimeters.

We are to prove Area $P >$ Area ABC.

Cons. Draw from C any line CD to AB. At C make $\angle DCH$ equal to the $\angle CDB$, and CH equal to DB, and join HD.

Dem. $\triangle CDH = \triangle CDB$. Why?

Hence Area $ABC = $ Area $ADHC$. Why?

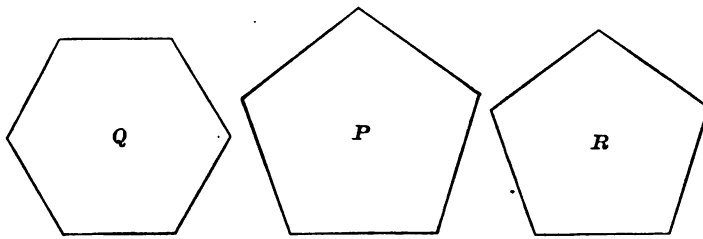
But Area $P >$ Area $ADHC$. Why?

Hence Area $P >$ Area ABC.

Similarly, P could be shown to be less than an isoperimetric regular pentagon, etc. Q.E.D.

616. The area of a circle is greater than the area of any polygon of equal perimeter.

617. Of all regular polygons having a given area, the perimeter of that which has the greatest number of sides is a minimum.



Post. Let Q and P be two regular polygons having equal areas, and Q having the greater number of sides.

We are to prove that the perimeter of P is greater than that of Q.

Dem. Let R be a regular polygon whose perimeter is equal to that of Q, but the number of sides the same as P.

Then $\qquad Q > R.$ $\qquad$ Why?

But $\qquad \text{Area } Q = \text{Area } P.$

$$\therefore \text{Area } P > \text{Area } R.$$

$$\therefore \text{Perimeter } P > \text{Perimeter } R. \qquad \text{Why?}$$

But $\qquad \text{Perimeter } R = \text{Perimeter } Q.$

$$\therefore \text{Perimeter } P > \text{Perimeter } Q.$$

Hence perimeter Q is a minimum. Q.E.D.

618. The circumference of a circle is less than the perimeter of any polygon of equal area.

619. The rectangle formed by the two segments of a line is maximum when the segments are equal.

PROBLEMS OF CONSTRUCTION.

620. In the demonstration of the foregoing theorems it has been *assumed* that certain constructions were possible; i.e. that perpendiculars and parallels could be drawn, that lines and angles could be bisected, etc. It is now proposed to show that those and many other problems *can be performed*, so our previous demonstrations are not vitiated that are in any way dependent upon such constructions.

621. The *solution* of a geometrical problem of construction involves in general *three steps;* viz.:

I. *The construction* proper, by the use of the compass and ruler.

II. *Demonstration* to prove the correctness of the construction.

III. *Discussion* of its limitations and applications, including the number of possible constructions.

If numerical or algebraical results are also required, then there is, of necessity,

IV. *Computation*, by which numerical values are ascertained, involving also the use of general symbols in obtaining algebraic formulæ.

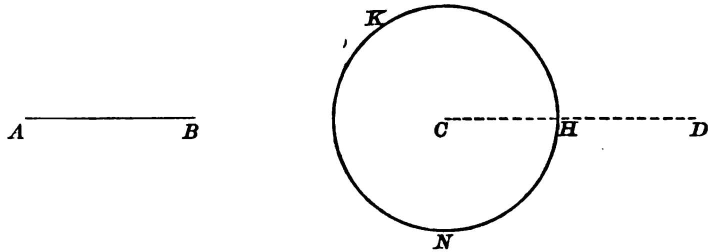
622. Each pupil should be provided with a good pair of compasses, for the use of either ink or pencil, a good ruler with *straight edges*, besides one hard and one soft pencil. The lead of the pencil should be sharpened *flat*, so that a fine line can be made.

PLANE PROBLEMS.

623. It is required to find a point which is a given distance from a given point.

Post. Let C be the given point, and AB the given distance.

We are required to find a point which shall be at the distance AB from C.



Cons. First, with the ruler, from C draw an indefinite straight line, as CD, in any direction. Then with the compasses, using C as a centre and AB as a radius, draw an arc cutting the line CD, as at H.

Then H is the required point; for,

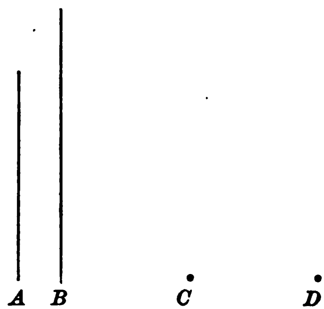
Dem. If, in applying the compasses to AB a circle had been constructed, and the circle HKN also completed, then the circles would have equal radii.

$\therefore$ H is the same distance from C that A is from B.

Discussion. Since, from the definition of a circle, all points in the circumference are equally distant from the centre, it follows that *every point* in the circumference HKN is the same distance from C that A is from B. Consequently *any point* in that circumference answers the conditions of the problem. Q.E.F.

623 (a). Whenever a line is found such that *any point* in it selected at random will fulfil certain specified conditions, or such that *all points* in it have a common property, that line is called the LOCUS of that point or points. Hence we may say, in the above case, that the circumference HKN is the *locus* of the point which is the distance AB from point C, or, as some prefer to put it, it is the locus of *all* points which are at the distance AB from point C.

624. It is required to find the point, having given its distances from two given points.



Post. Let the two given distances be A and B, and the two given points C and D.

We are required to find a point which is A distant from C, and B distant from D, or *vice versa.*

Sug. Find *locus* of the point which is A distant from point C, and locus of the point which is B distant from point D, and *vice versa.*

Dis. How many points, then, satisfy the condition of the problem?

Determine the result if

I. The distance between C and D had been greater than the sum of A and B.

II. If it had been equal to the sum of A and B.

III. If it had been equal to the difference of A and B.

IV. If it had been less than the difference of A and B. (See Theorem 229.)

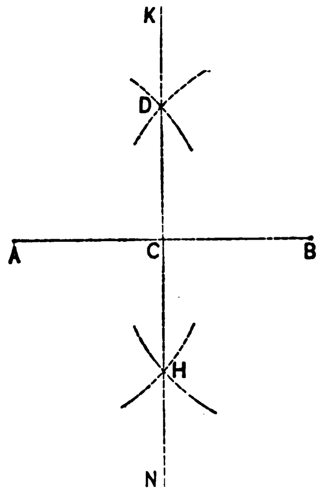
625. It is required to find the point which is equally distant from two given points.

Post. Let A and B be the two given points.

We are to find a point which is the same distance from A as from B.

It is evident that whether there be more or not, there must at least be one midway between *A* and *B*; so join *AB*.

With *A* and *B* as centres, construct, with the same radius, two circles which shall intersect. How can you tell whether or not they will intersect?



Connect the points of intersection, as *D* and *H*. Then *DH* sustains what relation to the two circles?

The line *AB* sustains what relation to the two circles?

Then what relation, as regards their position, between *AB* and *DH*?

How is the point *D* situated with reference to points *A* and *B*?

How is the point *H* situated with reference to the same points?

Then what must be the relation of *AC* and *CB* as regards magnitude? (Consult Theorem 82 or 230.)

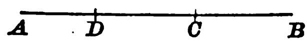
Suppose, now, *DH* be indefinitely extended, and any point in it selected at random. How will this point be situated with reference to the points *A* and *B*? Why?

What name, then, shall we give to the line *NK*? Why? Q.E.F.

626. It is required to bisect a given straight line.

Sug. Employ method similar to the previous one.

627. It is required to construct a perpendicular to a given straight line which shall pass through a given point in that line.



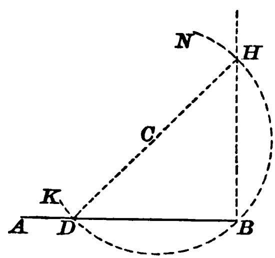
Post. Let AB be the given line, and C the given point.

We are required to construct a perpendicular to AB passing through point C.

Sug. Lay off equal distances each side of C, as CD and CB; then use Problem 625.

627 (*a*). It is required to construct a perpendicular to a line at one extremity.

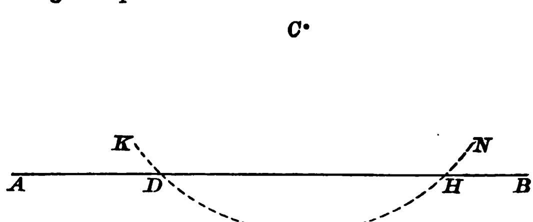
Sug. Extend the line; then use previous problem. In case the extension should not be possible or convenient, use the following.



Select any point at random, as C, making sure that it lies between A and B. Then, with CB as a radius, construct the circle or arc KBN. Through D, point where this circle intersects the given line, and C, draw the straight line DH, the point H being where this line intersects the arc KBN. Join HB.

Then HB is the perpendicular required. Why? Q.E.F.

628. It is required to construct a perpendicular to a given line from a given point without the line.



Post. Let AB be the given line, and C the given point.

Sug. With C as a centre, construct an arc which shall *intersect* AB.

Point C is how situated with reference to the two points of intersection D and H?

Can you find another point the same distance as C from those two points?

If that point and point C be joined by a straight line, how will that line be situated with reference to DH? (See Theorem 82.)

628 (a). It is required to bisect a given arc.

Sug. Connect the extremities of the given arc; then use Problem 626, and for proof consult Theorem 199.

629. It is required to bisect a given angle.

Sug. Use Problem 628 (a) and Theorem 192.

630. At a given point in a given line, it is required to construct an angle equal to a given angle.

Sug. Consult 191 and 192.

631. It is required to draw through a given point a line parallel to a given line.

Sug. Consult 87 and 84, and Problem 630.

632. Two angles of a triangle being given, it is required to construct the third angle.

Sug. Consult Theorems 58 and 98, and Problem 630.

633. Having given two sides of a triangle and their included angle, it is required to construct the triangle.

634. Having given two angles of a triangle and the side joining their vertices, it is required to construct the triangle.

635. Having given the three sides of a triangle, it is required to construct the triangle.

636. It is required to construct the locus of the point which is a given distance from a given straight line.

637. It is required to construct the locus of the point which is a given distance from a given circumference.

638. It is required to construct the locus of a point which is equally distant from two given parallel lines.

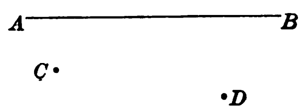
639. It is required to construct the locus of the point which is equally distant from two non-parallel lines in the same plane.

640. It is required to construct the locus of the point which is equally distant from the circumferences of two equal circles.

641. Having given the hypothenuse of a right triangle, it is required to construct the locus of the vertex of the right angle.

Sug. Consult Theorem 236 (*b*).

642. It is required to find in a given line, as *AB*, a point which is equally distant from two given points, as *C* and *D*.



Sug. Consult Problem 625.

643. It is required to find a point which is equally distant from three given points.

Sug. Join the points, then consult Problem 625.

644. Through a given point without a given line, it is required to draw a line which shall make an angle with the given line equal to a given angle.

645. It is required to construct the triangle, having given the base, the vertical angle, and one of the other angles.

646. It is required to construct the triangle, having given two sides and an angle opposite one of them.

647. It is required to find the centre of a given circumference or arc.

648. It is required to construct the circumference, having given three points in it.

649. It is required to construct a circumference which shall pass through the vertices of a given triangle.

650. It is required to find the locus of the centre of the circumference which shall pass through two given points.

651. With a given radius, it is required to construct the circle which shall pass through two given points.

652. It is required to construct the isosceles triangle, having given the base and the vertical angle.

653. It is required to construct a circumference which shall be a given distance from three given points.

654. It is required to construct a circle which shall have its centre in a given straight line and circumference passing through two given points.

655. It is required to find a point which shall be equally distant from two given points, and at a given distance from a third given point.

656. It is required to construct the equilateral triangle, having given one side.

657. It is required to trisect a right angle.

658. It is required to find a point which shall be equally distant from two given points, and also equally distant from two given parallel lines.

659. It is required to find a point which shall be equally distant from two given points, and also equally distant from two given non-parallel lines in the same plane.

660. It is required to find a point which shall be equally distant from two given parallel lines, and also equally distant from two non-parallel lines in the same plane.

661. It is required to construct

I.	an angle of 45°;	VI.	an angle of 75°;
II.	an angle of 60°;	VII.	an angle of 22° 30′;
III.	an angle of 30°;	VIII.	an angle of 52° 30′;
IV.	an angle of 15°;	IX.	an angle of 135°;
V.	an angle of 105°;	X.	an angle of 165°.

662. It is required to find a point in one side of a triangle which shall be equally distant from the other two sides.

663. It is required to find a point which shall be equally distant from two non-parallel lines in the same plane, and at a given distance from a given point.

664. It is required to construct the right triangle, having given the two legs.

665. It is required to construct the right triangle, having given the hypothenuse and one acute angle.

666. It is required to construct the right triangle, having given one leg and adjacent acute angle.

667. It is required to construct the right triangle, having given one leg and the acute angle opposite.

668. It is required to construct the right triangle, having given the hypothenuse and one leg.

669. It is required to construct a tangent to a given circle, having given the point of contact.

670. It is required to construct a tangent to a given circle, which shall pass through a given point outside the circle.

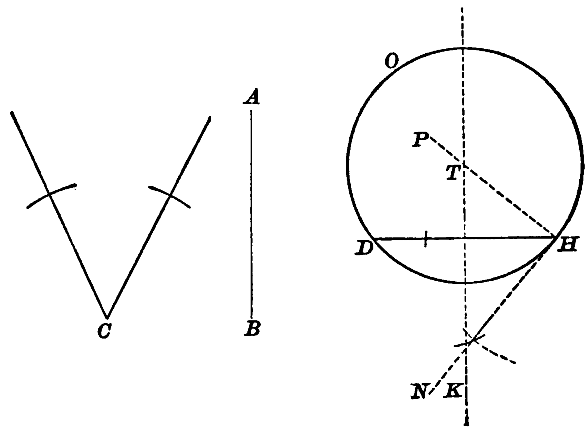
Sug. Connect the centre of the given circle with the given exterior point. On this line as a diameter construct a circle, and join the points of its intersection of the given circle with extremities of the diameter.

671. It is required to construct the parallelogram, having given two sides and their included angle.

672. It is required to construct a circle within a given triangle, so that the sides of the triangle shall be tangents of the circle.

Sug. Consult Theorems 117 and 125.

673. It is required to construct the circle, having given a chord and angle made by the chord and a tangent.



Let AB be the given chord, and C the angle made by the chord and tangent. Then one extremity of the chord must be the *point of contact.*

Take $DH = AB$. Construct angle DHN equal to angle C. Then H is the point of contact, and NH the tangent.

Find the *locus* of the centre of the circumference passing through D and H. (Problem 650.)

Then consult Theorem 202.

Hence T must be the centre of the circle required.

If from any point in the arc DOH (call the point Q) lines be drawn to D and H, how will the angle Q compare in magnitude with the angle DHN?

What is formed by the lines DH, DQ, and HQ?

If we call DH the base, what would you call the angle Q? What the point Q?

What might the arc DQH be named, then? Q.E.F.

674. Having given the base and vertical angle of a triangle, it is required to construct the locus of its vertex.

Sug. Consult the previous problem.

675. It is required to construct the triangle, having given the base, the vertical angle, and the altitude.

Sug. Use previous problem.

676. It is required to construct the triangle, having given the base, the vertical angle, and the median.

OPTIONAL PROBLEMS FOR ADVANCE WORK.

677. It is required to construct the triangle, having given the base, vertical angle, and perpendicular from one extremity of the base to the opposite side.

678. It is required to construct the isosceles triangle, having given the altitude and one of the equal angles.

679. It is required to construct the chord in a given circle, having given the middle point of the chord.

680. It is required to construct a circle whose circumference shall pass through the vertices of a given rectangle.

681. It is required to construct a tangent to a given circle which shall be parallel to a given straight line.

682. It is required to construct a tangent to a given circle which shall be perpendicular to a given straight line.

683. It is required to construct a tangent to a given circle which shall make a given angle with a given straight line.

684. It is required to construct the isosceles triangle, having given the vertical angle and a point in the base, in position.

685. It is required to construct the triangle, having given the altitude, the base, and an adjacent angle.

686. It is required to construct the triangle, having given the altitude, the base, and an adjacent side.

687. It is required to construct a rhombus, having given its base and altitude.

688. It is required to construct the triangle, having given the altitude and the sides which include the vertical angle.

689. It is required to construct the triangle, having given the altitude and angles adjacent to the base.

690. It is required to construct an isosceles triangle which shall have its vertical angle twice the sum of its other two angles.

691. It is required to construct the square, having given its diagonal.

692. It is required to construct the right triangle, having given the hypothenuse and perpendicular from the vertex of the right angle to the hypothenuse.

693. It is required to construct the locus of the centre of the circle of given radius tangent to a given straight line.

694. It is required to construct the circle of given radius which shall be tangent to two given non-parallel lines.

695. It is required to construct the circle of given radius which shall be tangent to a given straight line, and whose centre shall be in a given line not parallel to the former.

696. It is required to construct the circle which shall be tangent to a given line, and whose circumference shall pass through a given point.

697. It is required to find the locus of the centre of the circle of given radius which shall be tangent externally to a given circle.

698. It is required to construct the locus of the centre of the circle of given radius which shall be tangent internally to a given circle.

699. It is required to construct a circle which shall be tangent to a given line and a given circle.

700. It is required to construct a circle which shall be tangent to two given circles.

701. It is required to construct a circle which shall cut three equal chords of given length from three given non-parallel lines.

702. It is required to construct in a given circle a chord of given length passing through a given point.

703. It is required to construct in a given circle a chord of given length and parallel to a given straight line.

704. It is required to construct a line of given length passing through a given point between two given parallel lines.

705. It is required to construct a line of given length between two given non-parallel lines, and which shall be parallel to a given line.

706. It is required to construct a line of given length between two non-parallel lines, and which shall pass through a given point.

707. It is required to construct the right triangle, having given the hypothenuse and radius of the inscribed circle.

708. It is required to construct the right triangle, having given the radius of the inscribed circle and one acute angle.

709. It is required to construct the triangle, having given in position the middle points of its sides.

710. It is required to construct the triangle, having given the base, vertical angle, and radius of the circumscribing circle.

711. It is required to construct the isosceles triangle, having given the base and radius of the inscribed circle.

712. It is required to construct a straight line which shall pass through a given point and make equal angles with two given lines.

713. It is required to find a point in a given secant to a given circle such that the tangent to the circle from that point shall be of given length.

714. It is required to construct the right triangle, having given one leg and radius of the inscribed circle.

715. It is required to construct the right triangle, having given the median and altitude from the vertex of the right angle to the hypothenuse.

716. It is required to find the locus of the centre of the chord which passes through a given point in a given circle.

717. It is required to construct the triangle, having given the base, vertical angle, and sum of the other two sides.

718. It is required to construct a circle which shall be tangent to two given lines and at given point of contact in one.

719. It is required to inscribe a circle in a given sector.

720. It is required to construct a common tangent to two given circles.

Sug. Make five cases according to relative position of the circles. (See Theorem 229.)

721. It is required to inscribe a square in a given rhombus.

722. Having given two intersecting circles, it is required to draw a line through one of the points of intersection, so that the two intercepted chords shall be equal.

Sug. Join centres. Draw from point of intersection to middle of that line. From centres draw radii parallel to latter line. Through point of intersection draw line perpendicular to these radii.

723. It is required to construct an equilateral triangle having its vertices in three given parallel lines.

724. It is required to construct a tangent to a given circle with two given parallel secants so that the point of contact shall bisect the part between the secants.

725. It is required to construct three equal circles which shall be tangent to each other and also to a given circle externally.

726. It is required to construct three equal circles which shall be tangent to each other and also to a given circle internally.

727. It is required to construct three equal circles which shall be tangent to each other and also to the sides of an equilateral triangle.

728. It is required to construct a semicircle having its diameter in one of the sides of a given triangle and tangent to the other two sides.

729. It is required to construct a triangle, having given the radius of the inscribed circle and two sides.

730. It is required to find a point in a given line such that lines to that point from two given points without the line make equal angles with the line.

731. It is required to construct the triangle, having given the perimeter, altitude, and vertical angle.

Required Problems of Construction.

It is required,

732. To divide a given line into any number of equal parts.

Sug. From one extremity of the given line construct a line of indefinite length, making any convenient angle with the given line. Then, with any convenient unit of length assumed as a unit of measure, beginning at the vertex, lay off on the

indefinite line this unit as many times as it is required to divide the given line into parts. Then consult 316.

733. To divide a given line into parts proportional to any number of given lines.

Sug. Place the given parts so as to form one straight line, making any convenient angle with the line to be divided. Then consult 313 and 319.

734. To construct a line that shall be a *fourth* proportional to three given lines.

Sug. Consult Ratio and Proportion, 277, Theorem 313, and previous problem.

735. To construct a line that shall be a *third* proportional to two given lines.

Sug. Consult Ratio and Proportion, 279.

What must one of the given lines be if those two and one other are to form a proportion? Consider that in the construction.

736. To construct a mean proportional between two given lines.

Sug. Consult Theorem 367.

737. Given a polygon and an homologous side of another similar polygon, to construct the latter.

Sug. Consult Theorems 356 and 357.

738. To inscribe in a given circle a triangle similar to a given triangle.

739. To circumscribe about a given circle a triangle similar to a given triangle.

740. To construct a square which shall be equivalent to the sum of two given squares.

Sug. Consult Theorem 360.

741. To construct a square which shall be equivalent to the sum of three or more given squares.

Sug. Construct a square equivalent to two of the given squares, then one equivalent to that and one other of the given squares, and so on.

742. To construct a square which shall be equivalent to the difference of two given squares.

743. To construct a square equivalent to a given rectangle.

Sug. If x and y are the base and altitude of a rectangle, and n one side of an equivalent square, then

$$xy = n^2.$$

Whence $\quad x : n :: n : y;$ (See Theorem 294.)

or n is a mean proportional between x and y.

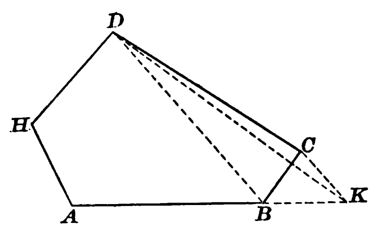
Hence consult Problem 736.

744. To construct a square which shall be equivalent to a given parallelogram.

745. To construct a square which shall be equivalent to a given triangle.

746. To construct a triangle which shall be equivalent to a given polygon of more than three sides.

Let *ABCDH* be a polygon of n sides.



Extend one of the sides, as *AB*, and construct the diagonal *DB*. Through vertex *C* draw *CK* parallel to *DB*, and join *DK*.

Considering *DB* the common base of the two triangles *DCB* and *DKB*, what is the relation between the areas of those two triangles? Why?

How does the area of the polygon *DKAH*, then, compare with that of *DCBAH*? Why?

How many sides has the former as compared with the latter?

Proceed in the same way with the polygon *DKAB*.

746 (*a*). To construct a square equivalent to any given polygon.

Sug. Use Problem 746, then 745.

746 (*b*). To construct a square equivalent to the sum of any number of given polygons.

746 (*c*). To construct a rectangle which shall be equivalent to a given square and the sum of whose base and altitude shall be equal to a given line.

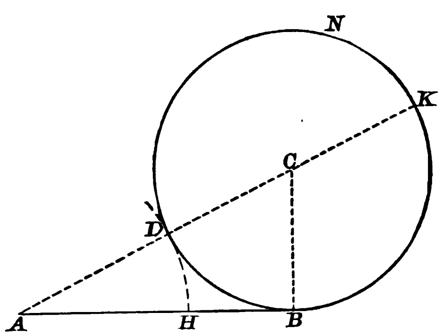
Sug. Upon the given line as a diameter construct a circle. Construct a line parallel to the diameter, distant from it one side of the given square. Then consult Theorem 367.

746 (*d*). To construct a rectangle which shall be equivalent to a given square and the difference of whose base and altitude shall be equal to a given line.

Sug. Proceed as in 746 (*c*), then at extremity of the diameter construct a tangent equal to one side of given square, and from other extremity of this tangent construct a secant through centre of circle. (Consult Theorem 370.)

747. To construct a right triangle which shall be equivalent to a given triangle and its hypothenuse equal to a given line.

748. To divide a given line into extreme and mean ratio. (See 280.)



Post. Let AB be the given line. At one extremity erect a perpendicular CB equal to one-half AB. With C as centre

and CB as a radius construct the circle DBN. Construct the secant AK passing through the centre C. With A as a centre and radius AD (point D being point of intersection of secant and circumference) construct the arc DH. Then the line AB will be divided in extreme and mean ratio at H.

Dem. Since $CB = \frac{AB}{2}$, $DK = AB$.

$AD : AB :: AB : AK$. Why?

$AD : AB - AD :: AB : AK - AB$. Why?

$AH : AB - AH :: AB : AK - DK$. Why?

$AH : HB :: AB : AD$. Why?

$AH : HB :: AB : AH$; Why?

or $HB : AH :: AH : AB$. Why?

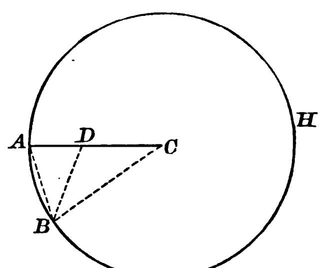
Hence the line AB is divided in extreme and mean ratio.

Q.E.F.

749. To inscribe a regular decagon in a given circle.

Post. Let ABH be the given circle, and AC one of its radii.

Consult Theorem 546, then use Problem 748.



Cons. With A as a centre and DC as a radius, construct chord AB. Join BC and BD.

Dem. $AC : AB :: AB : AD$. Why?

Hence the two $\triangle$ ABD and ABC are how related? (See Theorem 350.)

What *kind* of a triangle is ACB?

What kind of a triangle must ABD be, then?

What relation, then, between AB and DB? What between BD and DC?

What relation, then, between the $\angle$ CAB and CBA? Between the $\angle$ DAB and BAD? Between the $\angle$ BDA and CBA? Why? Between the $\angle$ DBC and DCB?

What relation does the $\angle BDA$ bear to the sum of DCB and DBC? Why?

What relation does $\angle DAB$ bear to $\angle C$, then?

Hence, what relation does $\angle ABC$ bear to $\angle C$?

Compare now $\angle DAB + \angle ABC$ with the $\angle C$, and finally compare $\angle DAB + \angle ABC + \angle C$ with the $\angle C$.

If the pupil has answered the above questions correctly, he will now have the equation,

$$\angle DAB + \angle ABC + \angle C = 5 \angle C.$$

What is the value of the first member of the above equation? Why?

Then $\qquad 5 \angle C = 2$ rt. $\angle$s;

or $\qquad 10 \angle C = 4$ rt. $\angle$s.

Whence $\qquad \angle C = \frac{1}{10}$ of 4 rt. $\angle$s.

Hence the arc AB is what part of the circumference?

$\therefore$ AB is the side of a regular inscribed decagon. Q.E.F.

750. To inscribe a square in a given circle.

Sug. Consult Theorem 513.

751. To inscribe a regular hexagon in a given circle.

Sug. Consult Theorem 480.

752. To inscribe a regular pentedecagon in a given circle.

Sug. Find the difference between a central angle of the regular decagon and that of a regular hexagon.

753. To inscribe in a given circle

I. a regular trigon;
II. a regular pentagon;
III. a regular octagon;
IV. a regular dodecagon;
V. a regular polygon of sixteen sides;
VI. a regular polygon of twenty sides.

754. To circumscribe around a given circle all the above-mentioned regular polygons.

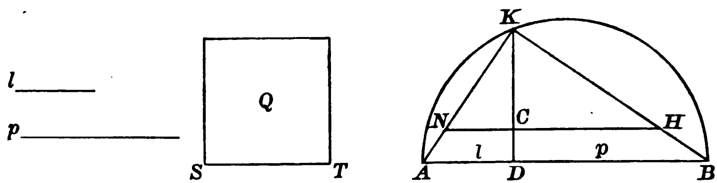
755. To inscribe in a given circle a regular polygon similar to a given regular polygon.

Sug. Construct a central angle equal to that of the given polygon, etc.

756. To circumscribe a circle about any given regular polygon.

757. Upon a given line as a base, to construct a rectangle equivalent to a given rectangle.

758. To construct a square whose ratio to a given square shall be the same as that of two given lines.



Post. Let Q be the given square, and l and p the two given lines.

We are required to construct a square (Q') so that

$$\text{Area } Q : \text{Area } Q' :: l : p.$$

Cons. Place the two given lines so as to form one straight line, as ADB.

On this as a diameter construct a semicircle, and at D erect the perpendicular DK. Join AK and BK.

Make KN equal to one side of the given square, as ST.

Draw NH parallel to AB. Then KH is the side of the required square.

Dem. $\overline{AK}^2 : \overline{BK}^2 :: AD : DB$. (Theorem 360, IV.)

Or, $\overline{AK}^2 : \overline{BK}^2 :: l : p$.

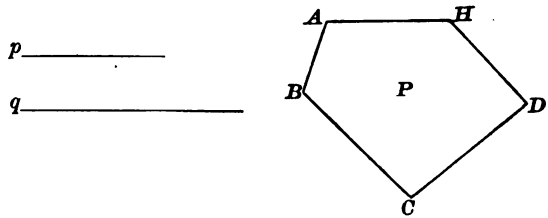
Again, $NK : HK :: AK : BK$. Why?

Hence $\overline{NK}^2 : \overline{HK}^2 :: \overline{AK}^2 : \overline{BK}^2$.

$\therefore \overline{NK}^2 : \overline{HK}^2 :: l : p$. Why?

Hence the square constructed on HK as a side is the required square. Q.E.F.

759. To construct a polygon similar to a given polygon, the ratio of whose areas shall be that of two given lines.



Post. Let $ABCDH$ be the given polygon (P), and p and q the two given lines.

We are required to construct a polygon (P') similar to P, so that

$$\text{Area } P : \text{Area } P' :: p : q.$$

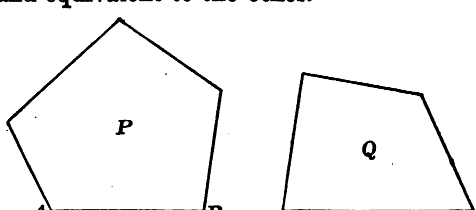
Cons. Upon any side of the polygon P, as AB, construct a square.

Then, by the previous problem, find the side of a square whose ratio to that of the square on AB shall be that of the two lines p and q.

Upon this line construct a polygon similar to polygon P. This will be the polygon required.

Dem. This will be left for the pupil.

759 (a). To construct a polygon similar to one of two given polygons and equivalent to the other.



Post. Let P and Q be the two given polygons. We are

required to construct a polygon similar to P and equivalent to Q.

Cons. Construct squares equivalent to each of the polygons P and Q.

Then find a fourth proportional to the sides of these squares and any side of the polygon P selected at random, as AB. Upon this fourth proportional as an homologous side construct a polygon similar to P. Then this polygon will be similar to P and equivalent to Q, and is therefore the polygon required.

Dem. This is also left for the pupil.

760. To construct upon a given line, as one side,

I.	a regular trigon;	V.	a regular pentagon;
II.	a regular tetragon;	VI.	a regular decagon.
III.	a regular hexagon;	VII.	a regular dodecagon;
IV.	a regular octagon;	VIII.	a regular pentedecagon.

761. To construct a regular hexagon, given one of its shorter diagonals.

762. To construct a regular pentagon, given one of its diagonals.

763. To construct a circle equivalent to the sum of two given circles.

764. To construct a circumference equal to the sum of two given circumferences.

765. To divide a given circle by a concentric circumference into two equal parts.

MISCELLANEOUS PLANE PROBLEMS FOR ADVANCE WORK.

I. TRIANGLES.

It is required to construct the triangle, having given,

766. Its base, vertical angle, and difference of the other two sides.

767. Its base, vertical angle, and a square which is equal to the sum of the squares upon the other two sides.

768. Its base, vertical angle, and a square which is equivalent to the difference of the squares upon the other two sides.

769. Its base, vertical angle, and sum of its altitude and the two remaining sides.

770. Its base, vertical angle, and the sum of its altitude and difference of the other two sides.

771. Its base, vertical angle, and difference between its altitude and sum of its other two sides.

772. Its base, vertical angle, and difference between its altitude and difference of the other two sides.

773. Its base, vertical angle, and ratio of its altitude to the sum of its other two sides.

774. Its base, vertical angle, and ratio of its altitude to the difference of its other two sides.

775. Its base, altitude, and sum of its other two sides.

776. Its base, altitude, and difference of its other two sides.

777. Its base, altitude, and ratio of the other two sides.

778. Its base, altitude, and a square equivalent to the rectangle of the other two sides.

779. Its base, altitude, and a square which is equivalent to the sum of the squares upon the other two sides.

780. Its base, altitude, and a square which is equivalent to the difference of the squares upon the other two sides.

781. Its vertical angle, sum of base and altitude, and sum of the other two sides.

782. Its vertical angle, sum of base and altitude, and difference of its other two sides.

783. Its vertical angle, sum of base and altitude, and ratio of the other two sides.

784. Its vertical angle, sum of base and altitude, and a square equivalent to the rectangle of its other two sides.

785. Its vertical angle, sum of base and altitude, and sum of the three sides.

786. Its vertical angle, sum of base and altitude, and difference between the base and sum of the other two sides.

787. Its vertical angle, sum of base and altitude, and difference between the base and difference of its other two sides.

788. Its vertical angle, sum of base and altitude, and the ratio of the base to the sum of the other two sides.

789. Its vertical angle, sum of base and altitude, and the ratio of the base to the difference of its other two sides.

790. Its vertical angle, altitude, and the square equivalent to the sum of the squares of the sides which include the vertical angle.

791. Its vertical angle, altitude, and radius of the circumscribing circle.

792. Its vertical angle, radius of the inscribed circle, and perimeter.

793. Its vertical angle, radius of the inscribed circle, and ratio of the sides including the vertical angle.

794. Its vertical angle, radius of the inscribed circle, and a square equivalent to the rectangle of the sum of the two sides including the vertical angle and the base.

795. Its vertical angle, radius of the inscribed circle, and a square whose area is equal to the difference between the sum of the squares of the sides including the vertical angle and the square of the base.

796. Its base, median, and sum of the other two sides.

797. Its base, median, and difference of the other two sides.

798. Its three altitudes.

799. Its three medians.

800. Two sides, and difference of the angles opposite them.

801. Its vertical angle, difference of the angles at the base, and difference of the other two sides.

802. Difference of the angles at the base, difference of the segments of the base made by the altitude, and sum of the other two sides.

803. It is required to construct the equilateral triangle, having given the three distances from its vertices to a common point within the triangle.

804. The same as 803, but the common point without the triangle.

II. QUADRILATERALS.

805. It is required to construct a square, having given

I. the sum of its diagonal and side.

II. the difference of its diagonal and side.

It is required to construct a rectangle, having given

806. The sum of two adjacent sides and its diagonal.

807. The difference of two adjacent sides and its diagonal.

808. One side, and sum of diagonal and adjacent side.

809. One side, and difference of diagonal and adjacent side.

It is required to construct the rhombus, having given

810. Its side and altitude.

811. Its altitude and lesser angle.

812. Its side and sum of its diagonals.

813. Its side and difference of its diagonals.

814. Its lesser angle and sum of its diagonals.

815. Its lesser angle and difference between its longer diagonal and altitude.

It is required to construct the rhomboid, having given

816. The longer side, sum of its diagonals, and larger angle made by the diagonals.

817. Its lesser angle, longer diagonal, and sum of two adjacent sides.

818. Its lesser angle, longer side, and sum of its altitude and lesser side.

819. Its lesser angle, shorter side, and difference of its longer diagonal and longer side.

It is required to construct an isosceles trapezoid, having given,

820. One leg, diagonal, and longer base.

821. Its longer base, diagonal, and lesser angle.

822. Its diagonal, altitude, and leg.

823. Its longer base, lesser angle, and sum of altitude and leg.

It is required to construct the trapezoid, having given

824. Its longer base, lesser angle (i.e. angle formed by longer base and a leg), and its diagonals.

825. Its longer base, one leg, lesser angle, and altitude.

826. Sum of its bases, the two legs, and lesser angle.

827. Difference of its bases, the two legs, and angle formed by its diagonals.

III. CIRCLES.

Having given a circle, it is required to construct

828. Three equal circles, tangent to the given circle externally, and tangent to each other.

829. Three equal circles, tangent to the given circle internally, and tangent to each other.

830. Four equal circles, as in (828) and (829).

831. Five equal circles, as in (828) and (829).

832. Six equal circles, as in (828) and (289).

833. A circle tangent to three given circles.

IV. TRANSFORMATION OF FIGURES.

It is required,

834. To transform a given triangle into an equivalent isosceles one having the same base.

835. To transform a given isosceles triangle into an equivalent equilateral one.

836. To transform a given triangle into an equivalent equilateral one.

837. To transform a given triangle into another equivalent triangle whose base and altitude shall be equal.

838. To transform a given triangle into another equivalent triangle, and similar to a given triangle.

839. To transform a given triangle into a triangle with one angle unchanged and its opposite side parallel to a given line.

840. To transform a given triangle into an equivalent triangle with a given perimeter.

841. To transform a triangle into a trapezoid, one of whose bases shall be the base of the triangle, and one of its adjacent angles one of the basal angles of the triangle.

842. To transform a given triangle into a right triangle with given perimeter.

843. To transform a given triangle into a parallelogram with given base and altitude.

844. To transform a parallelogram into a parallelogram with a given side.

845. To transform a parallelogram into a parallelogram having a given angle.

846. To transform a parallelogram into a parallelogram with given altitude.

To transform a square into

847. A right triangle.

848. An isosceles triangle.

849. An equilateral triangle.

850. A rectangle with given side.

851. A rectangle with given perimeter.

852. A rectangle with given difference of sides.

853. A rectangle with given diagonal.

To transform a rectangle into

854. A square.

855. An isosceles triangle.

856. An equilateral triangle.

857. A rectangle with given side.

858. A rectangle with given perimeter.

859. A rectangle with given difference of sides.

860. A rectangle with given diameter.

It is required to construct a parallelogram equivalent to the

a 861. Sum of two given parallelograms of equal altitudes.

862. Difference of two given parallelograms of equal altitudes.

863. Sum of two given parallelograms of equal bases.

864. Difference of two given parallelograms of equal bases.

865. Sum of two given parallelograms.

866. Difference of two given parallelograms.

It is required to transform a given parallelogram into

867. A triangle.

867 (*a*). An isosceles triangle.

868. A right triangle.

868 (*a*). An equilateral triangle.

869. A square.

870. A rhombus having for a diagonal one side of the parallelogram.

871. A rhombus having a given diagonal.

872. A rhombus having a given side.

873. A rhombus having a given altitude.

874. A parallelogram having a given side and diagonal.

875. To transform a rhombus into a square.

876. To inscribe in a given circle a rectangle equivalent to a given square.

To transform a trapezoid into

877. A triangle.

877 (*a*). A square.

878. A parallelogram having for one base the longer base of the trapezoid.

879. An isosceles trapezoid.

To transform a trapezium into

880. A triangle.

881. An isosceles triangle with given base.

882. A parallelogram.

883. A trapezoid with one side and the two adjacent angles unchanged.

V. DIVISION OF FIGURES.

884. It is required to divide a given triangle into any number of equivalent parts, by lines drawn from one vertex.

885. It is required to divide a given triangle into any number of equivalent parts, by lines drawn from any point in its perimeter selected at random.

886. It is required to divide a given triangle into any number of equivalent parts, by lines drawn from any point selected at random in the triangle.

887. It is required to divide a given triangle into any number of equivalent parts by lines parallel to one side.

888. It is required to divide a given triangle into any number of parts whose areas shall be in a given ratio, by lines drawn from one vertex.

889. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn from any point selected at random in the perimeter.

890. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn from any point selected at random in the triangle.

891. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn parallel to one side.

892. It is required to divide a given parallelogram into any

number of equal parts by lines drawn parallel to one pair of sides.

893. It is required to divide a given parallelogram into any number of parts, whose areas shall be in a given ratio, by lines parallel to one pair of sides.

894. It is required to divide a given parallelogram into two equivalent parts by a line drawn from any point selected at random in the perimeter.

895. It is required to divide a given parallelogram into two equivalent parts by a line drawn through any point in the parallelogram selected at random.

896. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from one vertex.

897. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from any point in the perimeter selected at random.

898. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from any point in the parallelogram selected at random.

899. It is required to divide a parallelogram into two equivalent parts by a line drawn parallel to a given line.

900. It is required to divide a parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn parallel to a given line.

901. It is required to divide a parallelogram into any number of equivalent parts by lines drawn from either vertex.

902. It is required to divide a parallelogram into any number of equivalent parts by lines drawn from any point in its perimeter selected at random.

903. It is required to divide a parallelogram into any num-

ber of equivalent parts by lines drawn from any point in the parallelogram selected at random.

904. It is required to divide a parallelogram into any number of equivalent parts by lines parallel to a given line.

905. It is required to divide a parallelogram into any number of parts, whose areas shall be in a given ratio, by lines parallel to a given line.

It is required to divide a trapezoid into two equivalent parts by a line drawn

906. Parallel to the bases.

907. Perpendicular to the bases.

908. Parallel to one of the legs.

909. Through one of its vertices.

910. Through a given point in one of its bases.

911. Through any point selected at random in its perimeter.

912. Through any point selected at random in the trapezoid.

913. Parallel to a given line.

It is required to divide a trapezoid into any number of equivalent parts by lines drawn

914. Parallel to the bases.

915. Perpendicular to the bases.

916. Parallel to one of its legs.

917. Through one of its vertices.

918. Through any point selected at random in one of its bases.

919. Through any point selected at random in its perimeter.

920. Through any point selected at random in the trapezoid.

921. Parallel to a given line.

It is required to divide a given trapezoid into any number of parts whose areas shall be in a given ratio by lines drawn

922. Parallel to the bases.

923. Perpendicular to the bases.

924. Parallel to one of the legs.

925. Through either vertex.

926. Through any point selected at random in one of the bases.

927. Through any point selected at random in its perimeter.

928. Through any point selected at random in the trapezoid.

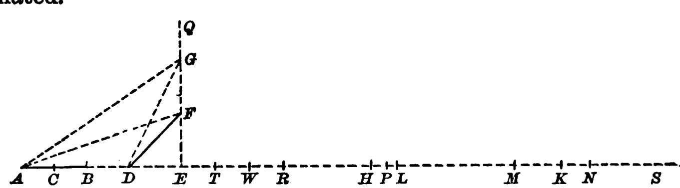
929. Parallel to a given line.

930. It is required to divide a trapezium into two equivalent parts by a line drawn from either vertex.

931. It is required to divide a trapezium into two equivalent parts by a line drawn from any point selected at random in its perimeter.

932. Given the diameter of a circle, it is required to construct a straight line equal in length to the circumference.

From Theorem 519 it is evident that it can only be approximated.



Let AB be the given diameter and C its middle point. Extend AB indefinitely as AS. Make BD and DE each equal to AB. At E erect the perpendicular EQ, and on it make EF and FG each equal to AB. Join AG, AF, DG, and DF. Lay off EH and HK each equal to AG, and from K lay off

KL equal to AF. Again, from L make LM equal to DG, and MN equal DF. Bisect EN at P; bisect EP at R; and then trisect ER at T and W. Then CT will be the required line approximately equal to the circumference of the circle whose diameter is AB; for, calling the diameter unity,

$$CE = 2\tfrac{1}{2},$$

$$EL = 2\,CH - KL = 2\sqrt{13} - \sqrt{10},$$

$$LM = \sqrt{5},$$

$$MN = \sqrt{2}.$$

$$\therefore\ EN = 2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}, \text{ and}$$

$$ET = \tfrac{1}{12}\,(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}), \text{ and therefore}$$

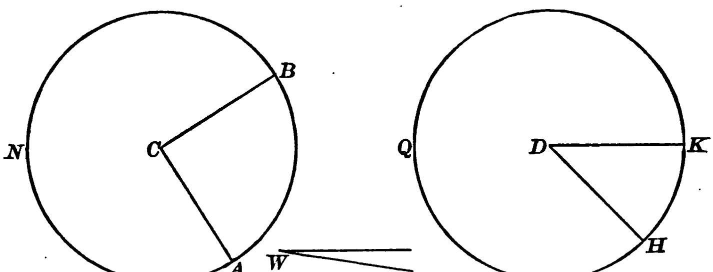
$$CT = 2\tfrac{1}{2} + \tfrac{1}{12}\,(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}) = 3.1415922 +.$$

Q.E.F.

APPENDIX.

1. The following theorem, and demonstrations of it, are given as a substitute for 234, for those teachers who may prefer it. Ratio and proportion (261), however, should be taken previous to attempting it.

2. In the same or equal circles two central angles are in the same ratio as the arcs which their sides intercept.



Post. Let NBA and QKH be two equal circles, and C and D two central angles.

We are to prove $\angle C : \angle D :: \text{Arc } AB : \text{Arc } HK$.

CASE I. — When the angles are *commensurable.*

Dem. If the angles are commensurable, there is some angle, as W, which will be contained an exact number of times in each.

Suppose it is contained m times in $\angle C$, and n times in $\angle D$.

Then $$\angle C : \angle D :: m : n.$$

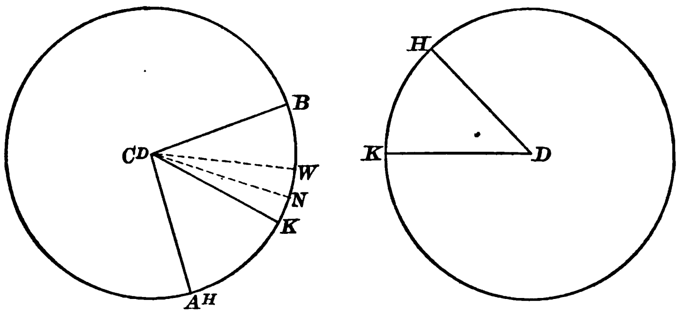
If, now, lines be drawn from C and D dividing the two angles into m and n equal parts respectively, each part being equal to angle W, then arc AB will be divided into m equal

arcs, and HK into n equal arcs, the divisions *all* being equal, by Theorem 191.

$\therefore$ Arc AB : Arc HK : : m : n.

$\therefore$ $\angle C$: $\angle D$: : Arc AB : Arc HK. (Theorem 288.)

CASE II. — When the angles are *incommensurable*.



Post. Let C and D be two incommensurable central angles in equal circles.

We are to prove $\angle C : \angle D$: : Arc AB : Arc HK.

Conceive them to be applied as in I.

Then if arc HK is not the fourth term of this proportion, some other arc greater or less than HK must be. Suppose it to be greater as AW.

Then $\angle ACB : \angle ACK$: : Arc AB : Arc AW. (*a*)

Now conceive the arc AB to be divided into equal parts by continued bisection until each part is less than KW. Then there must be at least one point of division between K and W, as N.

$\therefore$ $\angle ACB : \angle ACN$: : Arc AB : Arc AN. (*b*)

$\therefore$ $\angle ACK : \angle ACN$: : Arc AW : Arc AN.

(Theorem 300.)

But the arc AN is less than the arc AW. Consequently, if the proportion be a true one, the angle ACN must be less

than the angle ACK. On the contrary, it is greater, and therefore the proportion cannot be a true one. Therefore the supposition on which the argument was based; viz. that the fourth term must be an arc greater than HK, cannot be true. A similar argument will prove that it cannot be less.

Consequently it must be the arc HK.

$$\therefore \angle C : \angle D : : \text{Arc } AB : \text{Arc } HK. \qquad \text{Q.E.D.}$$

3. This really means the same thing as 234; viz. that the numerical measure of an angle at the centre of a circle is the same as the numerical measure of its intercepted arc, if the adopted unit of angle is the angle at the centre which intercepts the adopted unit of arc.

4. Another favorite method of demonstrating this theorem is that of the method of limits, hereto appended.

THEORY OF LIMITS.

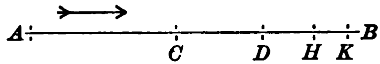
5. A *constant quantity,* or simply a *constant,* is a quantity whose value remains unchanged throughout the same discussion.

6. A *variable quantity,* or simply a *variable,* is a quantity which may assume different values in the same discussion, according to the conditions imposed.

7. The *limit* of a variable is a constant quantity, which the variable is said to *approach* in value whenever a *regular* and *definite increase* or *decrease* in value is assigned to the latter.

8. Whenever it can be shown that the value of a *variable,* by such constant increase or decrease in value, can be made to differ from that of its limit by less than any appreciable or assignable quantity, however small, this variable is said to *approach indefinitely to its limit.*

9. For example:



Suppose a point move from A toward B under the condition that during the first second it shall move over one-half the distance AB, or AC, and that during each successive second it shall move over one-half the remaining distance. Then at the end of the second second it would be at D, at the end of the third at H, at the end of the fourth at K, and so on. It is evident that it can never reach the point B, for there will constantly remain *one-half* the distance; but if its motion be continued indefinitely, it will approach indefinitely near to B.

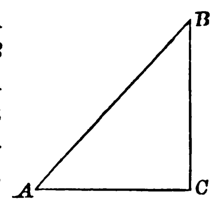
Consequently, the distance from A to the moving point is an *increasing variable*, and AB is its *limit;* while the distance from B to the moving point is a *decreasing variable*, with zero as its limit.

10. Other illustrations may be given; e.g.

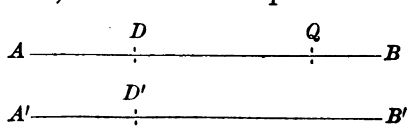
$$0.3333 + \cdots = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000} + \cdots.$$

Here the sum of the series of fractions is the increasing variable, and approaches $\frac{1}{3}$ as its limit.

Again: let ABC be a right triangle, with C the right angle, and consider the point B to move toward C; the angle A will then be a *decreasing variable* approaching zero as its limit, and the angle B will be an *increasing* variable approaching a right angle as its limit.



11. *Theorem.* If two variables are always equal, and each approaches a limit, their limits are equal.



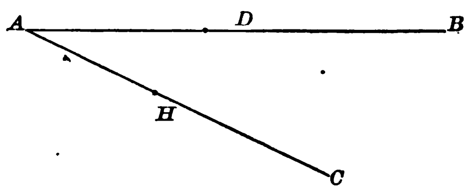
Post. Let AB and $A'B'$ be the limits to which the two equal variables AD and $A'D'$ indefinitely approach.

We are to prove $\qquad AB = A'B'$.

Dem. AB and $A'B'$ are either equal or unequal. Let us suppose them unequal, and that AB is the greater. Mark off AQ equal to $A'B'$.

Then the variable $A'D'$ cannot exceed $A'B'$, but the variable AD may exceed AQ, and consequently the variable AD becomes greater than the variable $A'D'$. This, however, is contrary to the hypothesis that the two variables must always be equal. Therefore AB and $A'B'$ cannot be unequal; i.e. they are equal. Q.E.D.

12. *Theorem.* **If two variables are in a constant ratio, their limits are in the same ratio.**



Post. Let AD and AH be two unequal variables, and approaching their respective limits AB and AC, and having a constant ratio such that $\dfrac{AD}{AH} = m$.

We are to prove that $\qquad \dfrac{AD}{AH} = \dfrac{AB}{AC}$,

or that $\qquad AD : AH :: AB : AC$.

Dem. Since $\quad \dfrac{AD}{AH} = m, \quad AD = m \times AH$.

And since $m \times AH$ will vary as AH varies, AD and $m \times AH$

are two equal variables, and therefore, from the previous theorem,

$$\text{Limit of } AD = \text{Limit of } m \times AH, \text{ or}$$

$$\text{Limit of } AD = m \times \text{Limit of } AH.$$

$$\therefore \frac{\text{Limit of } AD}{\text{Limit of } AH} = m;$$

but the limit of AD is AB, and the limit of AH is AC.

$$\therefore \frac{AB}{AC} = m;$$

and since $$\frac{AD}{AH} = m;$$

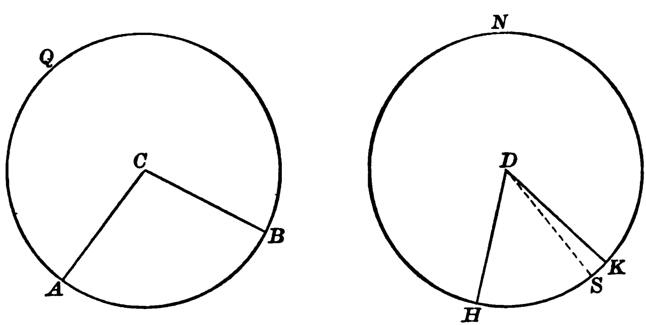
$$\therefore \frac{AB}{AC} = \frac{AD}{AH};$$

or $$AD : AH :: AB : AC.$$ Q.E.D.

13. Let us now apply these principles to the demonstration of the theorem enunciated at the beginning of the Appendix.

In Case I. the demonstration will remain unchanged.

Case II., when the arcs are *incommensurable.*



Post. Let AQB and NHK be two equal circles, and C and D two central angles whose arcs AB and HK are incommensurable.

We are to prove $$\frac{\angle C}{\angle D}=\frac{\text{Arc } AB}{\text{Arc } HK},$$

or $$\angle C : \angle D :: \text{Arc } AB : \text{Arc } HK.$$

Dem. Conceive the arc AB to be divided into any number of equal arcs, and one of these arcs to be applied as a unit of measure to the arc HK. It will be contained a certain number of times with a remainder, SK, less than the unit of measure.

Construct DS.

Then, since AB and HS are commensurable,

$$\frac{\angle C}{\angle D}=\frac{\text{Arc } AB}{\text{Arc } HS}.$$

If, now, the number of equal parts into which the arc AB is divided be indefinitely increased, the unit of measure of the arc HK will be correspondingly diminished, and the point S will get indefinitely near to K. Consequently the arc HS approaches indefinitely to HK, and the $\angle HDS$ to the $\angle HDK$.

Consequently the variables

$$\frac{\angle ACB}{\angle HDS} \text{ will approach its limit } \frac{\angle ACB}{\angle HDK},$$

and $$\frac{\text{Arc } AB}{\text{Arc } HS} \text{ will approach its limit } \frac{\text{Arc } AB}{\text{Arc } HK}.$$

But $$\frac{\angle ACB}{\angle HDS}=\frac{\text{Arc } AB}{\text{Arc } HS}.$$

$$\therefore \frac{\angle ACB}{\angle HDK}=\frac{\text{Arc } AB}{\text{Arc } HK},$$

because if two variables are always equal and approaching their limits, their limits are equal. Q.E.D.

14. Theorems 406 and 313 may be demonstrated by similar methods.

SYMMETRY.

15. Two points are said to be *symmetrical* with respect to a point when they are equidistant from, and in the same line with, this point. This point is called the *centre of symmetry.*

16. Two points are said to be *symmetrical* with respect to a line when the line that joins them is perpendicular to, and bisected by, this line. This line is called an *axis of symmetry.*

17. Two points are said to be *symmetrical* with respect to a plane when the line that joins them is perpendicular to, and bisected by, this plane. This plane is called a *plane of symmetry.*

18. The distance of either of two symmetrical points from the centre of symmetry is called the *radius of symmetry.*

19. Two plane figures are symmetrical with respect to a *centre, axis,* or a *plane,* when any point in either figure selected at random has a correspondingly symmetrical point in the other.

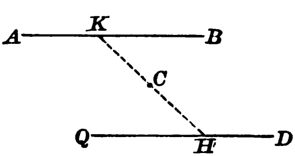


Fig. I.

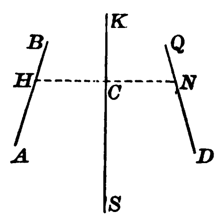


Fig. II.

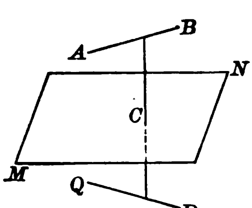


Fig. III.

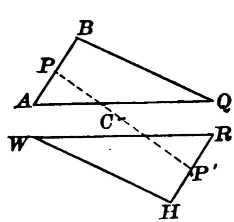


Fig. IV.

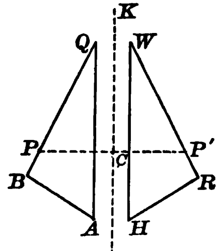


Fig. V.

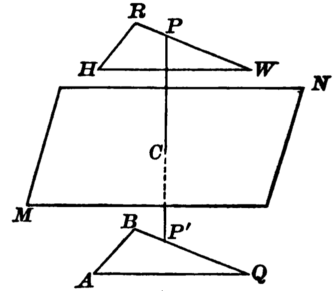


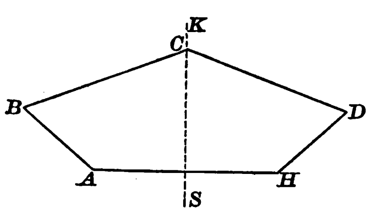
Fig. VI.

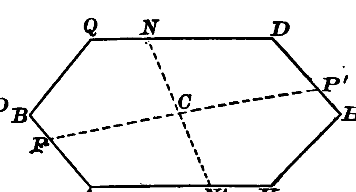
Thus, in Figs. I., II., and III., the lines *AB* and *QD* are symmetrical, with respect to the centre *C*, the axis *KS*, and the plane *MN*, respectively. In Figs. IV., V., and VI., the same is true of the triangles *ABQ* and *HRW*.

20. A plane figure is symmetrical

I. when it can be divided by an axis into two figures symmetrical with respect to that axis;

II. when it has a centre such that, if a line be drawn through it in any direction at random, the two points at which it intersects the perimeter are symmetrical with respect to that centre.





Thus figure *ABCDH* is symmetrical with respect to the axis *KS*, and *ABQDKH* with respect to the centre *C*. In the latter case *PP′* or *NN′* is called a *diameter of symmetry*. (See 18.)

21. A geometrical solid is symmetrical

I. when it can be divided by a plane into two solids symmetrical with respect to that plane;

II. when it has a centre such that, if a line be drawn through it in any direction selected at random, the two points at which it intersects the surface are symmetrical with respect to that centre.

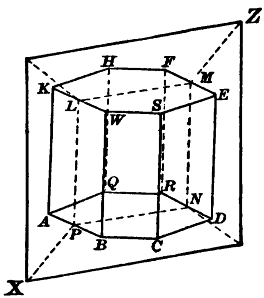


Fig. I.

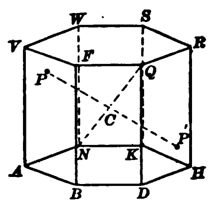


Fig. II.

Thus figure $ABCDEFHK$, Fig. I., is divided by the plane XZ at the lines LM, MN, NP, and PL, into two figures, symmetrical with respect to the plane XZ. Hence it is symmetrical.

Similarly, in Fig. II., the points P and P' being symmetrical with respect to the point C, according to above definition, the figure is symmetrical.

Theorems.

22. The centre of a circle is a centre of symmetry.

23. The diameter of a circle is an axis of symmetry.

24. The line which bisects the vertical angle of an isosceles triangle is an axis of symmetry.

25. Either altitude of an equilateral triangle is an axis of symmetry.

26. The point of intersection of two altitudes of an equilateral triangle is a centre of symmetry.

27. A segment of a circle is a symmetrical figure.

28. That part of a circle included between two parallel chords is a symmetrical figure.

29. The common part of two intersecting circles is a symmetrical figure.

30. The diagonal of a square is an axis of symmetry.

31. Every equilateral tetragon is a symmetrical figure.
(How many axes of symmetry does a square have?)

32. The point of intersection of the diagonals of a parallelogram is a centre of symmetry.

33. An isosceles trapezoid is a symmetrical figure.

34. If one diagonal of a tetragon divides it into two isosceles triangles, the other diagonal is an axis of symmetry.

35. The bisector of an angle of a regular polygon is an axis of symmetry.

36. The perpendicular bisector of one side of a regular polygon is an axis of symmetry.

37. If the angles at the extremities of one side of an equilateral pentagon be equal, the pentagon is a symmetrical figure.

38. If two diametrically opposite angles of an equilateral hexagon are equal, the hexagon is a symmetrical figure.

39. Every equiangular tetragon is a symmetrical figure.

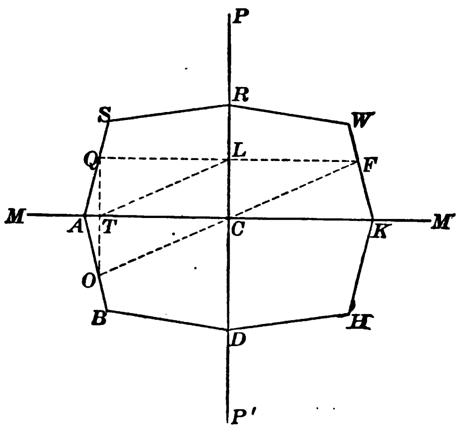
40. If a figure have two axes of symmetry perpendicular to each other, their intersection is a centre of symmetry.

Post. Let $ABDH$, etc., be a figure having the two axes of symmetry PP' and MM' $\perp$ to each other.

We are to prove that their point of intersection C is a centre of symmetry.

Cons. From any point in the perimeter selected at random, as Q, construct $QO \perp MM'$, $QF \perp PP'$, and join TL, OC, and FC.

Dem. $OT = TQ = LC$. Why?



Then what kind of a figure is $OTLC$? What relation, then, between TL and OC? Compare in a similar manner TL and CF. Finally compare OC and CF.

Hence points O and F are in the same line with and equidistant from C. Hence C is a centre of symmetry. Q.E.D.